Efficient Allocation with Informational Externalities*

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Abstract

We consider a seller of an item who faces potential buyers whose valuations depend on multiple signals. The seller has the ability to control the order in which buyers' signals arrive, but cannot observe these signals directly. It is known from the literature that when there are informational externalities and signals arrive all at once efficiency is unattainable. We show that by designing the order in which signals arrive, the seller can attain efficiency even in the presence of informational externalities.

1 Introduction

A seller of an item often faces buyers whose valuations of the object are functions of multiple signals. Examples include cases where the valuations buyers assign to the object are functions of different physical properties of the item, such as a gas company that has to check different properties of an oil field, or a buyer of a house who needs to run several checks regarding the quality of the house's infrastructure. In these situations the seller has the ability to control the set of signals that buyers are exposed to and the timing in which these signals arrive by deciding which tests will be conducted and in

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what order. However, in many situations the seller cannot observe these signals directly either because the tests are carried out privately, or because she lacks the knowledge to infer how a result of a test translates to a value of a signal. In this paper we show that the seller can exploit her ability to schedule buyers' private signals in order to attain efficiency in settings where efficiency could not be attained had the buyers learned their private signals prior to the sale.

As an illustration we present the following example. Consider a government that owns an oil field. The government can either use this field independently or sell it to an oil company that would then hold a monopoly in the market. The firm and only the firm can conduct two tests. One test reveals the firm's marginal cost and the other reveals its fixed cost. The government for its part can decide which tests will be conducted and in what order. The demand function is commonly known and so both the firm's profit and the consumer surplus can be deduced from any result of the tests. The government wants to implement an efficient sale, that is, to sell the field if and only if the value of the social welfare as a result of the sale is greater than its value in the case of no sale. The social welfare in the case of no sale would be some constant known to the government.[1] The social welfare in the case of a sale would be the sum of the monopoly's profit and the consumer surplus. Our results show that the government's possibility of implementing the efficient sale depends on the selling procedure it uses. If the government begins the selling process after the firm has conducted both tests then efficiency is not attainable. However, the government can attain efficiency if it uses the following sequential mechanism. First, it lets the firm conduct the test that reveals its marginal cost. Then it offers the firm a menu of options, and each option provides the firm with the right to buy the field at a specified price.[2] Lastly, it lets the firm conduct the second test and offers the field to the firm at the specified price.

In the first part of the paper we consider the case of a single buyer and a single seller. We look at environments in which the buyer receives two payoff-relevant signals in a sequential manner. We characterize implementability in terms of the relationship between the buyer's valuation and the decision rule. We first reestablish the result of

[1]The assumption that the social welfare in the case of no sale does not depend on the company's private information is not necessary for the result, and is assumed for ease of exposition.

[2]These options are priced such that an option to buy the field at a lower specified price costs more.

Maskin (1992), Dasgupta and Maskin (2000), and Jehiel and Moldovanu (2001) that in static environments (i.e., environments where each buyer knows all of his signals) a decision rule is implementable if and only if it is monotonic with respect to the buyer's valuation. This means that a necessary condition for implementation is that the buyer's valuation does not change along the decision rule's boundary.[3] We then show that a decision rule is implementable in a sequential environment if and only if it is monotonic with respect to each of the buyer's signals and, in addition, the buyer's valuation moves monotonically along the decision rule's boundary.

We use these results to compare the possibility of efficient implementation between static and sequential environments. For this purpose we examine the effect of the buyer's information on the social welfare. In situations where the effect of the buyer's information on the social welfare is limited to its effect on the buyer's value, efficiency is attainable in both static and sequential environments. This is because the boundary of the efficient decision rule coincides with one of the buyer's isovalue curves. By contrast, in situations where the buyer's information has other externalities on the social welfare, efficiency is typically unattainable in static environments. This is because the boundary of the efficient decision rule does not coincide with any of the buyer's isovalue curves. Nonetheless, efficiency can be attained in sequential environments. This happens in cases where the ratio between the effects of the first and the second signals is greater with respect to the social welfare than with respect to the buyer's valuation. In such cases the buyer's valuation is monotonic along the boundary of the efficient decision rule and efficiency is attainable. The above example illustrates such a case. The firm's information about its marginal cost affects the social welfare not only by its effect on the firm's profits but also by its effect on the consumer's surplus. On the other hand, the firm's information about its fixed cost affects the social welfare and the firm's profit to the same extent. Therefore, if the government exposes the firm first to the signal that reveals its marginal cost and then to the signal that reveals its fixed cost, the above condition about the ratio of the signals' effects holds. Hence, efficiency is attainable.

In order to further characterize the settings in which sequential mechanisms provide a higher expected social welfare than static mechanisms we also consider the case

[3] The boundary of the decision rule is the boundary between the set of signals that maps "do not sell" and the set of signals that maps "do sell."

where the efficient decision rule cannot be implemented even in sequential environments. When the buyer's valuation moves monotonically along the efficient decision rule's boundary, efficient sequential implementation is possible if and only if the buyer's signals arrive in a particular order. We show that when the buyer's signals do not arrive in the right order, the second-best decision rule of the sequential environment provides the same expected social welfare as the second-best decision rule of the static environment. When the buyer's valuation is not monotonic along the boundary of the efficient decision rule, we present sufficient conditions for the second-best decision rule of the sequential environment to provide a higher expected social welfare than the second-best decision rule of the static environment.

In the second part of the paper we extend our analysis to settings with multiple buyers. We introduce to sequential environments a notion of implementation that is robust to others' signal distributions. Our analysis focuses on settings of multiple buyers with multidimensional signals and interdependent valuations.[4] The possibility of efficient static implementation in these settings has been analyzed in several important papers. Jehiel and Moldovanu (2001) extend the insight of Maskin (1992) and Dasgupta and Maskin (2000) to show that if buyers' signals are independent then efficient Bayesian implementation is generically impossible.[5] In particular, this result holds in the setting of linear valuations. Our first result shows that when signals arrive sequentially efficiency is no longer generically impossible in the linear valuations setting. That is, we show that efficiency can be implementable by sequential mechanisms on a set of valuations of a positive measure.

Bikhchandani (2006) analyzes a static setting of multiple buyers with multidimensional signals and interdependent valuations, where the agents' valuations satisfy a certain single-crossing property. Bikhchandani focuses on a class of mechanisms that satisfy conditional efficiency, i.e., mechanisms that allocate the item to a buyer only if his valuation is higher than all the other buyers' valuations. Bikhchandani characterizes the most efficient mechanism out of the ex-post implementable mechanisms in

[4]In environments of private values, efficiency can be attained in static settings; see Vickrey (1961), Clarke (1971), and Groves (1973). Bergemann and Valimaki (2010) and Athey and Segal (2012) propose mechanisms that implement efficient allocations in dynamic settings with private values.

[5]The setting in Jehiel and Moldovanu (2001) is more general and contains the setting of the allocation of an indivisible good as a private case.

this class.[6] As implied by Jehiel and Moldovanu's (2001) result, such a mechanism is generically inefficient.[7] Our second result shows that it is generally possible to achieve better allocations in sequential environments. That is, we show that there exists an implementable, conditionally efficient sequential mechanism that is more efficient than the mechanism offered by Bikhchandani.

There are several other works that present positive results on efficient implementation in environments of interdependent valuations and multidimensional signals. Mezzeti (2004) shows that in settings where it is possible to condition transfers on realizations of payoffs, efficiency can be attained in static environments by executing the following two-stage mechanism. In the first stage agents report their signals and the efficient allocation is chosen. In the second stage each agent observes his payoff and reports it, and each agent receives a transfer that is equal to the sum of the other agents' reported payoffs. Our results show that in sequential environments it is possible to achieve efficiency even in settings that require both the allocation and transfers to depend only on agents' signals.[8] Johnson et al. (2003) show that in static environments where buyers' signals are correlated such that different values of an agent's signal imply different distributions of the other agents' signals, efficient Bayesian implementation is (approximately) possible.[9] Their result depends on the aforementioned stochastic relevance assumption and on the assumption that the distribution of signals is commonly known. Our results apply to more general environments. Specifically, the environments we consider require less restrictive demands on the behavior of the joint distribution (in particular, agents' signals can be independent) and buyers need not have complete knowledge of the joint distribution. The methods to attain efficiency in static settings that appear in both the aforementioned papers have been extended to dynamic settings.

[6] Jehiel et al. (2006) consider the possibility of ex-post implementation in general environments of interdependent valuations and multidimensional signals. They show that it is generically impossible to ex-post implement non-trivial deterministic decision rules. Bikhchandani's result shows that Jehiel et al.'s result does not apply to private-good settings, i.e., settings where agents care only about their own personal allocation (private-good settings are negligible in Jehiel et al.).

[7] The loss of efficiency occurs because incentive compatibility implies that the seller must keep the item on a set of signals of a positive measure and gains from trade are not realized.

[8] In many situations the relizations of payoffs occur long after the allocation is executed. In these situations it seems impractical to condition the mechanisms on payoff realizations.

[9] The applicability of this result is problematic if agents' signals are almost independent since then transfers become arbitrarily large.

Liu (2014) uses correlation among types and He and Li (2016) use transfers that are contingent on payoff realizations, to induce efficiency in dynamic settings.

Another strand of literature deals with the question of how the seller can exploit a sequential arrival of signals to improve her revenue. Courty and Li (2000) analyze a revenue-maximizing mechanism in the setting of a single seller and a single buyer, where the buyer's signals arrive in two periods and the seller can execute the selling mechanism in the first period. Eső and Szentes (2007) analyze a revenue-maximizing mechanism in the case of multiple buyers where the seller can decide whether the buyers will be exposed to additional signals. They show that the seller finds it optimal to release all the information to the buyers. In this case the buyers do not receive any information rent on those additional signals. That is, the seller's expected revenue from the optimal mechanism is equal to the revenue she would receive from the optimal mechanism in the case where she knows these signals' realizations.

The rest of the paper is organized as follows. In Section 2 we discuss the single-buyer case. We characterize the sets of implementable decision rules in both static and sequential environments, and present examples in which the efficient decision rule belongs to the latter set but not to the former. In Section 3 we discuss the case of multiple buyers. Section 4 concludes. Proofs are relegated to the appendices.

2 Single Buyer

We start our analysis with the case where the seller is facing a single potential buyer. We characterize the set of implementable decision rules in both static and sequential environments. The necessary and sufficient conditions we derive for the sequential case are more permissive than the ones derived for the static case. An implication of this result is that in sequential environments, unlike in static environments, efficiency can be attained even in the presence of informational externalities. We apply this observation to privatization processes and show how governments can benefit from applying sequential mechanisms in these processes. Lastly, we consider the case where efficiency is not attainable even by sequential mechanisms. We show that in some settings the second-best solution provided by sequential mechanisms is equivalent to the solution provided by static mechanisms, and that in some settings they provide a higher expected social welfare. We extend the results of this section to the case of multiple buyers in the following section.

2.1 Setup

Consider a seller of a single item facing a potential buyer. There are two periods 1 and 2. The buyer receives a private signal $\theta^1 \in [0,1]$ in period 1, and a private signal $\theta^2 \in [0,1]$ in period 2. The signals θ^1 and θ^2 are independent, θ^2 is uniformly distributed, and this is common knowledge.[10] The buyer's valuation V is a function of his signals $V : [0,1]^2 \to \mathbb{R}_+$. We assume that V is continuously differentiable and strictly increasing in θ^1 and θ^2. The buyer's payoff is minus his payment to the seller, plus, in case he gets the item, the value of the item. We denote by A the set of feasible allocations $A = \{0,1\}$, where 1 is the allocation that assigns the item to the buyer, and 0 is the allocation that assigns the item to the seller. A *decision rule* is a function[11] $q : \Theta \to A$. A *social choice function,* s, assigns an allocation and a payment to the seller for every realization of signals, i.e., $s(\theta) = (q(\theta), t(\theta))$, where $q(\theta) \in A$ and $t(\theta) \in \mathbb{R}$.

2.2 Static Mechanism

We start with an analysis of static mechanisms. Static mechanisms are mechanisms that are activated after the buyer has been exposed to both his signals θ^1 and θ^2. We restrict our attention to direct mechanisms and show in Appendix A that all our results still hold in the set of indirect mechanisms. We say that a social choice function $(q(\theta), t(\theta))$ is *implementable in a static mechanism* if for every (θ^1, θ^2) we have

$$\left(\theta^1, \theta^2\right) \in \underset{\left(\hat{\theta}^1, \hat{\theta}^2\right) \in [0,1]^2}{\arg\max} V\left(\theta^1, \theta^2\right) \cdot q\left(\hat{\theta}^1, \hat{\theta}^2\right) - t\left(\hat{\theta}^1, \hat{\theta}^2\right)$$

We say that a decision rule $q(\theta)$ is *implementable in a static mechanism* if there exists a transfer function $t(\theta)$ such that $(q(\theta), t(\theta))$ is implementable in a static mechanism.

[10] In Section 2.2.1 we show that our analysis can be extended to any distribution function by a proper modification of the assumption on the valuation function.

[11] In Appendix A we discuss the implications of our focus on deterministic mechanisms.

We now characterize the set of implementable decision rules in a static setting. The necessary and sufficient conditions for implementability are that the decision rule maps according to the buyer's valuations and that the decision rule is monotonic with respect to these valuations.

Claim 1. A decision rule $q(\theta)$ is ex-post implementable in a static mechanism if and only if it is of the following form:

$$q(\theta) = \begin{cases} 1 \text{ if} & V(\theta^1, \theta^2) > C \\ 0 \text{ or } 1 \text{ if} & V(\theta^1, \theta^2) = C \\ 0 & \text{otherwise} \end{cases}$$

for some $C \in \mathbb{R}$.

The argument of the proof is as follows. We first show that the relevant information for the decision rule is the buyer's valuation. Implementability implies that the buyer pays one price if he wins the item, $t(1)$, and another price if he does not win the item, $t(0)$. Let θ and θ' be two pairs of signals on the same buyer's isovalue curve such that $q(\theta) = 1$ and $q\left(\theta'\right) = 0$; then implementability implies that

$$V(\theta) - t(1) \geq -t(0)$$

and

$$V\left(\theta'\right) - t(1) \leq -t(0)$$

and so

$$V(\theta) = V\left(\theta'\right) = t(1) - t(0)$$

That is, there can be at most one isovalue curve for which two pairs of signals that lie on this isovalue curve are assigned with different alternatives. This means that the decision rule maps alternatives according to the buyer's valuation. Therefore, the problem is equivalent to implementation with respect to a unidimensional signal, where the monotonicity of the decision rule with respect to the valuation is necessary and sufficient for implementation.

2.3 Sequential Mechanisms

We proceed to analyze sequential mechanisms. Sequential mechanisms are mechanisms that are activated in two periods. We restrict our attention to direct mechanisms and show in Appendix A that all our results still hold in the set of indirect mechanisms. In the first period the buyer is asked to report his type θ^1. Then in the second period the buyer is asked to report his type θ^2. We say that a social choice function $(q(\theta), t(\theta))$ is *implementable in a sequential mechanism* if the following conditions hold:

1.
$$E_{\theta^2}\left[V\left(\theta^1,\theta^2\right)\cdot q\left(\theta^1,\theta^2\right)-t\left(\theta^1,\theta^2\right)\right]\geq$$
$$E_{\theta^2}\left[V\left(\theta^1,\theta^2\right)\cdot q\left(\hat{\theta}^1,\hat{\theta}^2\left(\theta^2\right)\right)-t\left(\hat{\theta}^1,\hat{\theta}^2\left(\theta^2\right)\right)\right]$$
for every $\theta^1 \in [0,1]$ and $\hat{\theta}^1 \in [0,1]$ and every $\hat{\theta}^2 : [0,1] \to [0,1]$.

2.
$$V\left(\theta^1,\theta^2\right)\cdot q\left(\theta^1,\theta^2\right)-t\left(\theta^1,\theta^2\right)\geq$$
$$V\left(\theta^1,\theta^2\right)\cdot q\left(\theta^1,\hat{\theta}^2\right)-t\left(\theta^1,\hat{\theta}^2\right)$$
for every $(\theta^1,\theta^2) \in [0,1]^2$ and $\hat{\theta}^2 \in [0,1]$.

Condition 2 implies that given a truthful report of θ^1, reporting θ^2 truthfully is optimal for the buyer. Condition 1 implies that reporting θ^1 truthfully is optimal for the buyer for any subsequent report about θ^2. We say that a decision rule $q(\theta)$ is *implementable in a sequential mechanism* if there exists a transfer function $t(\theta)$ such that $(q(\theta), t(\theta))$ is implementable in a sequential mechanism.

2.3.1 General Distributions of Signals

We assumed that the buyer's signals θ^1 that θ^2 are independent, that θ^2 is uniformly distributed, and that the buyer's valuation function is strictly increasing in θ^1 and θ^2. We now show that, by an appropriate modification of the assumption on the buyer's valuation function, our analysis can be applied to any distribution of the buyer's signals. Assume that θ^1 and θ^2 are distributed according to a distribution function F. Building on an observation of Esö and Szentes (2007), we define $\tilde{\theta}^2 := F(\theta^2|\theta^1)$. Given any θ^1 the conditional distribution of $\tilde{\theta}^2$ is uniform on the interval $[0,1]$:

$$P_r\left(\tilde{\theta}^2 \leq X|\theta^1\right) = P_r\left(\theta^2 \leq F^{-1}\left(X|\theta^1\right)|\theta^1\right) = F\left(F^{-1}\left(X|\theta^1\right)|\theta^1\right) = X$$

That is, the signals θ^1 and $\tilde{\theta}^2$ are independent and $\tilde{\theta}^2$ is uniformly distributed on the interval $[0,1]$. We define $\tilde{V}\left(\theta^1, \tilde{\theta}^2\right) := V\left(\theta^1, F^{-1}\left(\tilde{\theta}^2|\theta^1\right)\right) = V(\theta^1, \theta^2)$; i.e., any restriction that we impose on the valuation function in the independent case we apply to the function $\tilde{V}$ through the appropriate restrictions on the original valuation function[12] V. In this case the problem of implementing a decision rule $q\left(\theta^1, \tilde{\theta}^2\right)$ falls into our analysis.

2.3.2 Characterization of the Set of Implementable Decision Rules

A sequential mechanism is composed of two mechanisms, one in each period. The mechanism in the first period requires the buyer to report his first-period type and accordingly sets the properties of the second-period mechanism. The mechanism in the second-period requires the buyer to report his second-period type and accordingly decides whether the buyer will receive the item. A property of the second-period mechanism is that it sets a single price for the item, and so the set of alternatives offered in the first mechanism is the set of optional prices for the item in the second period. We now show that a decision rule is implementable in a sequential environment if and only if it assigns lower prices in the second period to higher types in the first period.

Consider the set $\mathtt{C} := \{C : [0,1] \to [0,1] \text{ s.t. } C \text{ is decreasing}\}$. For each $C \in \mathtt{C}$ we denote $\underline{\theta}^{1,C} := \inf\{\theta^1 \text{ s.t. } C(\theta^1) < 1\}$ and $\overline{\theta}^{1,C} := \sup\{\theta^1 \text{ s.t. } C(\theta^1) > 0\}$.

Theorem 2. *A decision rule $q(\theta)$ is implementable in a sequential mechanism if and*

[12]For example, Courty and Li (2000) consider the case where $V\left(\theta^1, \theta^2\right) = \theta^2$. In this case $\tilde{V}\left(\theta^1, \tilde{\theta}^2\right) = F^{-1}\left(\tilde{\theta}^2|\theta^1\right)$. Applying the demand of monotonicity on $\tilde{V}$ translates to the demand that $f\left(\theta^2|\theta^1\right)$ is strictly decreasing in θ^1.

only if there exists a function $C \in \mathtt{C}$ *such that*

$$q(\theta) = \begin{cases} 1 \ \textit{if} & \theta^2 > C(\theta^1) \\ 0 \ \textit{or} \ 1 \ \textit{if} & \theta^2 = C(\theta^1) \\ 0 & \textit{otherwise} \end{cases}$$

and in addition $V(\theta^1, C(\theta^1))$ *is a decreasing function of* θ^1 *in the segment*[13] $\left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]$.

The argument of the proof is as follows. Assume the buyer reports his first type θ^1 truthfully. Then in the second period we are facing an implementation problem with respect to a unidimensional signal θ^2, where the buyer's valuation is $V(\theta^1, \theta^2)$. Since V is strictly monotone in θ^2, implementability holds if and only if the decision rule is monotonic with respect to θ^2. The threshold is set at $C(\theta^1)$ and the payment to the seller in case of a sale is[14]

$$\tau\left(\theta^1\right) := \begin{cases} V(\theta^1, C(\theta^1)) & \text{if } \underline{\theta}^{1,C} \leq \theta^1 \leq \overline{\theta}^{1,C} \\ V\left(\overline{\theta}^{1,C}, 0\right) & \text{if } \overline{\theta}^{1,C} < \theta^1 \leq 1 \end{cases}$$

This implies that each report of θ^1 in the first period sets a price for the item in the second period. In addition, the buyer is charged a fee $p(\theta^1)$ for participating in the mechanism that sets the price $\tau(\theta^1)$. Thus, the transfer function is set as

$$t(\theta) = \begin{cases} p(\theta^1) + \tau(\theta^1) & \text{if } \ q(\theta) = 1 \\ p(\theta^1) & \text{if } \ q(\theta) = 0 \end{cases}$$

We now show why the property that $\tau(\theta^1)$ is decreasing is necessary and sufficient for implementation to take place.

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[13] For every $\theta^1 < \underline{\theta}^{1,C}$ we have $C\left(\theta^1\right) = 1$ and $q\left(\theta^1, 1\right) = 0$. For every $\overline{\theta}^{1,C} < \theta^1$ we have $C\left(\theta^1\right) = 0$ and $q\left(\theta^1, 0\right) = 1$. That is, for these θ^1 the decision rule $q\left(\theta^1, \cdot\right)$ is a constant function.

[14] In the segment $\left[0, \underline{\theta}^{1,C}\right)$ the threshold type is 1 and the buyer will not receive the item.

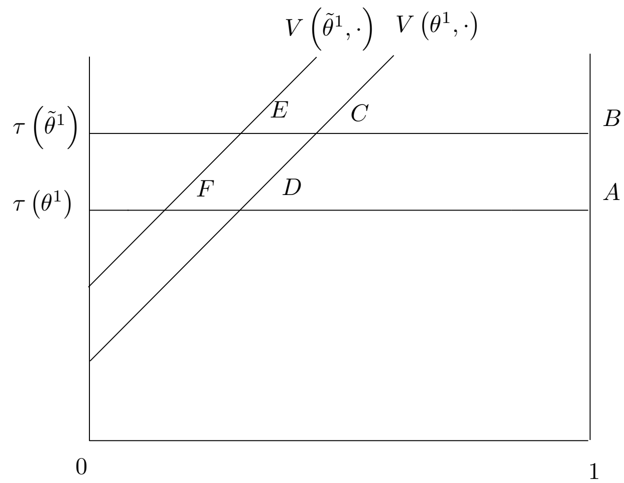


Figure 1: Necessity

We start by showing why the property that $\tau\left(\theta^1\right)$ is decreasing is necessary. Consider a buyer of type θ^1 in the first period who is facing a price τ in the second period. The buyer will decide to buy the object in the second period if and only if his valuation is higher than the price. This means that the expected utility of the buyer is equal to the integral of $V\left(\theta^1, s\right) - \tau$ from $V^{-1}\left(\theta^1, \tau\right)$ to 1. Assume two types $\theta^1 < \tilde{\theta}^1$ such that $\tau\left(\theta^1\right) < \tau\left(\tilde{\theta}^1\right)$. These types' valuation functions and prices are depicted in Figure 1. If type $\tilde{\theta}^1$ deviates and reports θ^1, then type $\tilde{\theta}^1$'s expected utility increases in the size of the area of the trapezoid ABEF. Therefore, in order to prevent such a deviation, the difference $p\left(\theta^1\right) - p\left(\tilde{\theta}^1\right)$ must be greater than or equal to the area of the trapezoid ABEF. If type θ^1 deviates and report $\tilde{\theta}^1$ then then type θ^1's expected utility decreases in the size of the area of the trapezoid ABCD, but he gains $p\left(\theta^1\right) - p\left(\tilde{\theta}^1\right)$ in the participation fee. Therefore, such a deviation is worthwhile for θ^1 and implementation fails.

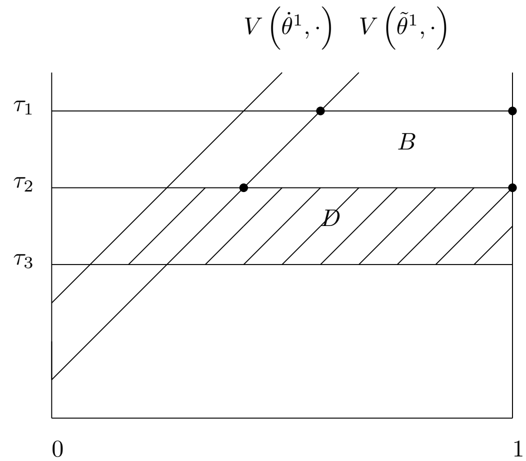


Figure 2: Sufficiency

We now show why the property that $\tau(\theta^1)$ is decreasing is sufficient. We illustrate by an example how to construct participation fees $p(\theta^1)$ that support the sorting of the first period's types in the case where $\tau(\theta^1)$ is decreasing. Consider the following mechanism. There are three options: payment τ_1 for types $\left[0,\tilde{\theta}^1\right]$, payment τ_2 for types $\left(\tilde{\theta}^1,\dot{\theta}^1\right)$, and payment τ_3 for types $\left[\dot{\theta}^1,1\right]$. This mechanism is depicted in Figure 2. The following participation fees allow for the implementation of this mechanism:[15]

$$p\left(\theta^1\right)=\begin{cases}0 & \text{if } \theta^1\in\left[0,\tilde{\theta}^1\right]\\ B & \text{if } \theta^1\in\left(\tilde{\theta}^1,\dot{\theta}^1\right)\\ B+D & \text{if } \theta^1\in\left[\dot{\theta}^1,1\right]\end{cases}$$

Payment B is type $\tilde{\theta}^1$'s willingness to pay for moving from τ_1 to τ_2. Therefore, all types smaller than $\tilde{\theta}^1$ strictly prefer τ_1 and a participation fee of zero to τ_2 and a participation fee of B, while all types larger than $\tilde{\theta}^1$ strictly prefer τ_2 and a participation fee of B to τ_1 and a participation fee of zero. Payment D is type $\dot{\theta}^1$'s willingness to pay for moving from τ_2 to τ_3. Therefore, all types smaller than $\dot{\theta}^1$ strictly prefer τ_2 and a participation fee of B to τ_3 and a participation fee of $B+D$, while all types larger

[15]Payment B equals the area of the trapezoid whose vertices are marked in black dots. Payment D equals the area of the trapezoid marked by the diagonal lines.

than $\dot{\theta}^1$ strictly prefer τ_3 and a participation fee of $B + D$ to τ_2 and a participation fee of B. These preferences are transitive and therefore the mechanism is implementable.

To conclude, an implementable sequential mechanism provides the buyer in the first period with a menu of options, each sets a strike price for the item in the second period. All types of the buyer agree on the ordinal order of the options (the lower the price in the second period, the better). However, they differ in the intensity of their preferences, such that higher θ^1 types are more willing to pay for better options. To achieve implementation, higher types must be assigned better options and hence $\tau(\theta^1)$ must be decreasing.

2.4 Implementation of Efficient Decision Rules

This paper considers a seller whose objective is to attain efficiency, namely, who is looking to execute the allocation that would produce the greatest social welfare. In the context of a single buyer the seller will want to sell the item if and only if the social welfare in the case where the buyer owns the item is greater than it would be if the seller kept the item. In situations where the seller's valuation is independent of the information that is privately known to the buyer, efficiency is attainable in static environments.[16] When the seller's valuation depends on the buyer's private information the possibility of attaining efficiency in static settings depends on the dimensionality of the information. When the buyer's information is unidimensional efficiency is attainable if the buyer's valuation satisfies a single-crossing condition.[17] If, however, the buyer's information is multidimensional efficiency is typically unattainable.[18] In this subsection we show that in sequential environments efficiency can be attained even in the latter case.

[16]This is done by simply setting the price to be the seller's valuation.

[17]See, for example, Maskin (1992), Dasgupta and Maskin (2000), and Perry and Reny (2002).

[18]See, Maskin (1992), Dasgupta and Maskin (2000) and Jehiel and Moldovanu (2001).

2.4.1 Necessary and Sufficient Conditions for Attaining Efficiency

We consider efficient decision rules that take the following form. There exists a function $U : [0,1]^2 \to \mathbb{R}$ such that

$$q(\theta) = \begin{cases} 1 \text{ if} & U(\theta^1, \theta^2) > C \\ 0 \text{ or } 1 \text{ if} & U(\theta^1, \theta^2) = C \\ 0 & \text{otherwise} \end{cases}$$

where $C \in \mathbb{R}$ and $U(\theta^1, \theta^2)$ is continuously differentiable and strictly increasing in θ^1 and θ^2. We define the set $\bar{U}$ to be the boundary of the efficient decision rule, i.e., $\bar{U} := \{(\theta^1, \theta^2) \text{ s.t. } U(\theta^1, \theta^2) = C\}$. Since static implementation requires that the decision rule map according to the buyer's valuation, we reach the following conclusion:

Corollary 3. *The decision rule $q(\theta)$ is implementable by a static mechanism if and only if for every*

$$\left(\tilde{\theta}^1, \tilde{\theta}^2\right), \left(\theta^1, \theta^2\right) \in \bar{U}$$

we have

$$V\left(\tilde{\theta}^1, \tilde{\theta}^2\right) = V\left(\theta^1, \theta^2\right)$$

In words, the boundary of the decision rule coincides with one of the buyer's isovalue curves. We denote by $[\underline{u}, \overline{u}]$ the segment of all θ^1 such that there exists θ^2 where $(\theta^1, \theta^2) \in \bar{U}$. We define $\tilde{\theta}^2(\theta^1)$ to be the function that assigned to any $\theta^1 \in [\underline{u}, \overline{u}]$ the threshold type it inflicts with respect to θ^2, i.e., $\tilde{\theta}^2(\theta^1) := \theta^2$ s.t. $(\theta^1, \theta^2) \in \bar{U}$. In a sequential environment the price for the item in the second period for every such $\theta^1 \in [\underline{u}, \overline{u}]$ is $\tau(\theta^1) := V\left(\theta^1, \tilde{\theta}^2(\theta^1)\right)$. Since sequential implementation requires $\tau(\theta^1)$ to be decreasing, we reach the following conclusion:

Corollary 4. *The decision rule $q(\theta)$ is implementable by a sequential mechanism if and only if for every*

$$\left(\dot{\theta}^1, \tilde{\theta}^2\left(\dot{\theta}^1\right)\right), \left(\tilde{\theta}^1, \tilde{\theta}^2\left(\tilde{\theta}^1\right)\right) \in \bar{U}$$

such that $\tilde{\theta}^1 < \dot{\theta}^1$ *we have*

$$V\left(\dot{\theta}^1, \tilde{\theta}^2\left(\dot{\theta}^1\right)\right) \leq V\left(\tilde{\theta}^1, \tilde{\theta}^2\left(\tilde{\theta}^1\right)\right)$$

In words, the buyer's valuation weakly decreases as we move rightward along the boundary of the decision rule.

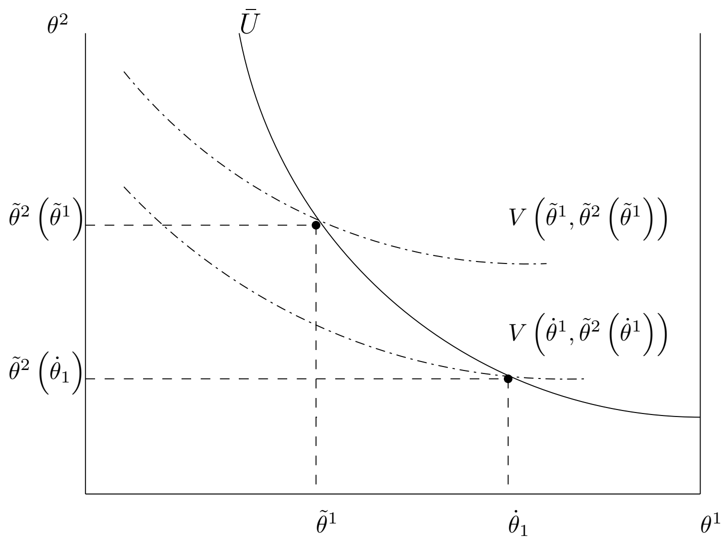


Figure 3: The buyer's valuation decreases along the boundary of the decision rule.

The boundary of the decision rule coincides with one of the buyer's isovalue curves if and only if the marginal rates of substitutions of $V(\theta)$ and $U(\theta)$ are equal for every $(\theta^1, \theta^2) \in \bar{U}$. The buyer's valuation weakly decreases as we move rightward along the boundary of the decision rule if and only if the marginal rates of substitutions of $V(\theta)$ are weakly smaller than those of $U(\theta)$ for every $(\theta^1, \theta^2) \in \bar{U}$. Therefore, the above corollaries can be presented in terms of marginal rates of substitutions. The decision rule $q(\theta)$ is implementable by a static mechanism if and only if

$$\frac{\partial V / \partial \theta^1}{\partial V / \partial \theta^2}\left(\theta^1, \theta^2\right) = \frac{\partial U / \partial \theta^1}{\partial U / \partial \theta^2}\left(\theta^1, \theta^2\right)$$

for every $(\theta^1, \theta^2) \in \bar{U}$, and is implementable by a sequential mechanism if and only if

$$\frac{\partial V / \partial \theta^1}{\partial V / \partial \theta^2}\left(\theta^1, \theta^2\right) \leq \frac{\partial U / \partial \theta^1}{\partial U / \partial \theta^2}\left(\theta^1, \theta^2\right)$$

for every $(\theta^1, \theta^2) \in \bar{U}$.

Remark. Consider the case where $\frac{\partial V / \partial \theta^1}{\partial V / \partial \theta^2}(\theta^1, \theta^2) \geq \frac{\partial U / \partial \theta^1}{\partial U / \partial \theta^2}(\theta^1, \theta^2)$; this inequality is equivalent to $\frac{\partial V / \partial \theta^2}{\partial V / \partial \theta^1}(\theta^1, \theta^2) \leq \frac{\partial U / \partial \theta^2}{\partial U / \partial \theta^1}(\theta^1, \theta^2)$. Therefore, if the seller has the ability to control the order in which signals arrive, she can achieve efficiency by setting signal θ^2 to arrive first and θ^1 to arrive second. Put differently, if the seller controls the order of signals, then the monotonicity of the buyer's valuation along the efficient decision rule's boundary is necessary and sufficient for implementation.

2.4.2 Application: Privatization Processes

Perhaps the most common and important situation of a seller who is interested in executing an efficient sale is privatization. The following examples illustrate how activating sequential mechanisms can assist the government in executing efficient privatization.

Example 5. Consider a government that owns an asset. The government can either use this asset independently and in this case the social welfare would be C, or it can sell it to a private firm that would then be a monopoly in the market. The government wants to sell the asset to the firm if and only if the social welfare as a result of the sale will be greater than C. The cost function of the monopoly depends on two arguments: an argument θ^1 that affects its marginal cost (as θ^1 increases the marginal cost decreases) and an argument θ^2 that affects its fixed cost (as θ^2 increases the fixed cost decreases). The demand function is common knowledge. The profit function of the monopoly, $V(\theta^1, \theta^2)$, is strictly increasing in both arguments. If the firm buys the asset, the social welfare, $U(\theta^1, \theta^2)$, will be the sum of the monopoly's profit and the consumer surplus and it is also strictly increasing in both arguments.[19] The first-best decision rule of the

[19]Transfers from the monopoly do not affect the social welfare.

government is

$$q(\theta) = \begin{cases} 1 \text{ if} & U(\theta^1, \theta^2) > C \\ 0 \text{ or } 1 \text{ if} & U(\theta^1, \theta^2) = C \\ 0 & \text{otherwise} \end{cases}$$

Now, since $U(\theta^1, \theta^2)$ takes into account the positive effect of the reduction of the marginal cost on both the firm's profit and the consumer surplus we have that ${}^{\partial V}/_{\partial \theta^1}(\theta^1, \theta^2) < {}^{\partial U}/_{\partial \theta^1}(\theta^1, \theta^2)$. The consumer surplus is not affected by a change in the fixed cost and therefore ${}^{\partial V}/_{\partial \theta^2}(\theta^1, \theta^2) = {}^{\partial U}/_{\partial \theta^2}(\theta^1, \theta^2)$. We conclude that $\frac{\partial V/\partial \theta^1}{\partial V/\partial \theta^2}(\theta^1, \theta^2) < \frac{\partial U/\partial \theta^1}{\partial U/\partial \theta^2}(\theta^1, \theta^2)$. Hence the efficient rule is implementable by a sequential mechanism and is not implementable by a static mechanism.

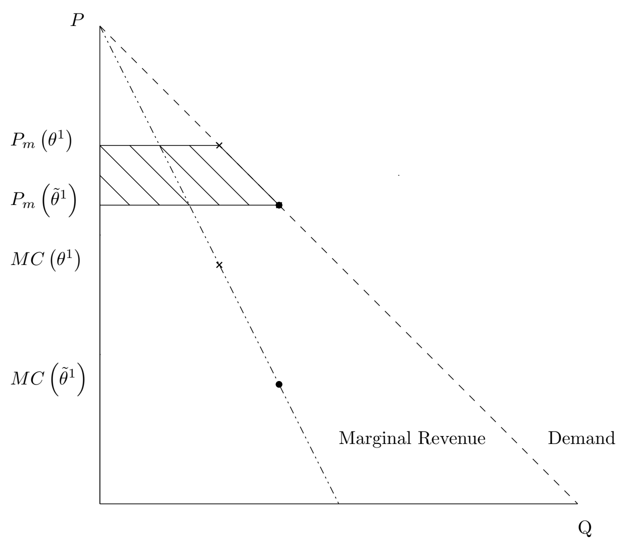


Figure 4: The marginal cost decreases by moving from θ^1 to $\tilde{\theta}^1$, where $\theta^1 < \tilde{\theta}^1$; as a result the monopoly price decreases and the consumer surplus increases.

Example 6. Consider a government that owns an asset that can be allocated to one of two potential buyers, 1 and 2. Buyer 1 and buyer 2 can both conduct tests that affect their valuations of the asset, but while the signals of buyer 2 become public the signals of buyer 1 are private.[20] The signals of buyer 1, (θ^1, θ^2), affect the valuation of both buyers

[20]For example, buyer 2 can be a government office that is obligated by law to disclose its signals.

in a linear manner.[21] The valuation function of buyer 1 is $V_1(\theta^1, \theta^2) = \alpha^1\theta^1 + \alpha^2\theta^2 + b$ and the valuation function of buyer 2 is $V_2(\theta^1, \theta^2) = \beta^1\theta^1 + \beta^2\theta^2 + g$, where $g > b$ and $\alpha^k > \beta^k > 0$ for $k \in \{1, 2\}$. We define $U(\theta^1, \theta^2) := (\alpha^1 - \beta^1)\theta^1 + (\alpha^2 - \beta^2)\theta^2$. The first-best decision rule of the government is

$$q(\theta) = \begin{cases} 1 \text{ if} & U(\theta^1, \theta^2) > g - b \\ 0 \text{ or } 1 \text{ if} & U(\theta^1, \theta^2) = g - b \\ 0 & \text{otherwise} \end{cases}$$

By Corollary 3 the first-best decision rule is implementable in a static environment if and only if $\frac{\partial V_1/\partial \theta^1}{\partial V_1/\partial \theta^2}(\theta^1, \theta^2) = \frac{\partial U/\partial \theta^1}{\partial U/\partial \theta^2}(\theta^1, \theta^2)$ for every $(\theta^1, \theta^2) \in \bar{U}$, namely, if and only if $\frac{\beta^1}{\beta^2} = \frac{\alpha^1}{\alpha^2}$. This condition is met only for a set of parameters of measure zero in $\mathbb{R}^4_+$, and so efficiency is generically impossible. By Corollary 4 the first-best decision rule is implementable in a sequential environment if and only if $\frac{\partial V_1/\partial \theta^1}{\partial V_1/\partial \theta^2}(\theta^1, \theta^2) \leq \frac{\partial U/\partial \theta^1}{\partial U/\partial \theta^2}(\theta^1, \theta^2)$ for every $(\theta^1, \theta^2) \in \bar{U}$, namely, if and only if $\frac{\beta^1}{\beta^2} \leq \frac{\alpha^1}{\alpha^2}$. This condition is met for a set of parameters of positive measure in $\mathbb{R}^4_+$. This means that if the government can control the timing in which the tests of buyer 1 are conducted, then it can implement an efficient sale in scenarios where such a sale cannot be attained if buyer 1 learned his signals prior to the sale. If, moreover, the government can control the order in which these tests are conducted, then it can always attain efficiency. This is because the condition on the marginal rates of substitutions is always satisfied with respect to some order of signal arrival.

2.4.3 Second-best Analysis

In the previous subsubsections we saw that applying sequential mechanisms can enable the seller to attain efficiency in situations where efficiency cannot be attained by static mechanisms. The necessary and sufficient condition for efficient implementation in a sequential environments is translated to a condition on the variations of the rates of substitutions of the buyer's valuation with respect to the boundary of the efficient decision rule. In this subsubsection we consider the case where this condition does not

[21]The assumption of linearity is not essential for the result and is assumed for simplicity.

hold and analyze whether the use of sequential mechanisms can still enhance the social welfare.[22] We first deal with the case where the buyer's valuation is increasing as we move rightward along the boundary of the efficient decision rule. We show that in this case the social welfare cannot increase from applying sequential mechanisms.

Theorem 7. *Assume that* $\frac{\partial V/\partial\theta^1}{\partial V/\partial\theta^2}(\theta^1,\theta^2) > \frac{\partial U/\partial\theta^1}{\partial U/\partial\theta^2}(\theta^1,\theta^2)$ *for every* $(\theta^1,\theta^2) \in \bar{U}$*. Then there exists a second-best decision rule with the property that* $\tau(\theta^1) = \tau$*, where* $\tau \in \mathbb{R}$*; namely, it sets a single price in the second period.*

The argument of the proof is as follows. We show that there exists a second-best sequential mechanism in which there exists $\tilde{\theta}^1 \in [\underline{u}, \overline{u}]$ for which $\tau\left(\tilde{\theta}^1\right) = V\left(\tilde{\theta}^1, \tilde{\theta}^2\left(\tilde{\theta}^1\right)\right)$. That is, the boundary of the second-best decision rule intersects with the boundary of the efficient decision rule.[23] Let us consider such a mechanism. For every $\theta^1 < \tilde{\theta}^1$, sequential implementability implies that $\tau(\theta^1) \geq \tau\left(\tilde{\theta}^1\right)$. For any such θ^1, if $\tau(\theta^1) > \tau\left(\tilde{\theta}^1\right)$, then $\{\theta_2 \text{ s.t. } V(\theta^1,\theta^2) \geq \tau(\theta^1)\}$, the set of signals for which a sale is executed, is strictly contained in $\left\{\theta_2 \text{ s.t. } V(\theta^1,\theta^2) \geq \tau\left(\tilde{\theta}^1\right)\right\}$, the set of signals for which a sale would have been executed if the price had been $\tau\left(\tilde{\theta}^1\right)$. Since the MRS of V is steeper than the MRS of U, these two sets are contained in $\{\theta_2 \text{ s.t. } U(\theta^1,\theta^2) \geq C\}$, the set of signals for which a sale should be executed according to the efficient decision rule. Therefore, if $\tau(\theta^1) > \tau\left(\tilde{\theta}^1\right)$, then the set of signals where a sale does not happen but should happen increases with respect to the case where $\tau(\theta^1) = \tau\left(\tilde{\theta}^1\right)$ and the expected social welfare decreases. For every $\theta^1 > \tilde{\theta}^1$ sequential implementability implies that $\tau(\theta^1) \leq \tau\left(\tilde{\theta}^1\right)$. For any such θ_1, if $\tau(\theta^1) < \tau\left(\tilde{\theta}^1\right)$ then $\{\theta^2 \text{ s.t. } V(\theta^1,\theta^2) \geq \tau(\theta^1)\}$ strictly contains the set $\left\{\theta^2 \text{ s.t. } V(\theta^1,\theta^2) \geq \tau\left(\tilde{\theta}^1\right)\right\}$ and since the MRS of V is steeper than the MRS of U these two sets contain the set $\{\theta^2 \text{ s.t. } U(\theta^1,\theta^2) \geq C\}$. Therefore, if $\tau(\theta^1) < \tau\left(\tilde{\theta}^1\right)$ then the set of signals where a sale does happen but should not happen increases with respect to the case where $\tau(\theta^1) = \tau\left(\tilde{\theta}^1\right)$ and the expected social welfare decreases. We conclude that there exists a second-best mechanism in which there is a single price in the second period. Such a mechanism is also implementable in a static environment; hence, the social welfare cannot increase from applying sequential mechanisms.

[22] We still restrict our analysis to deterministic mechanisms.

[23] The proof appears in the appendix.

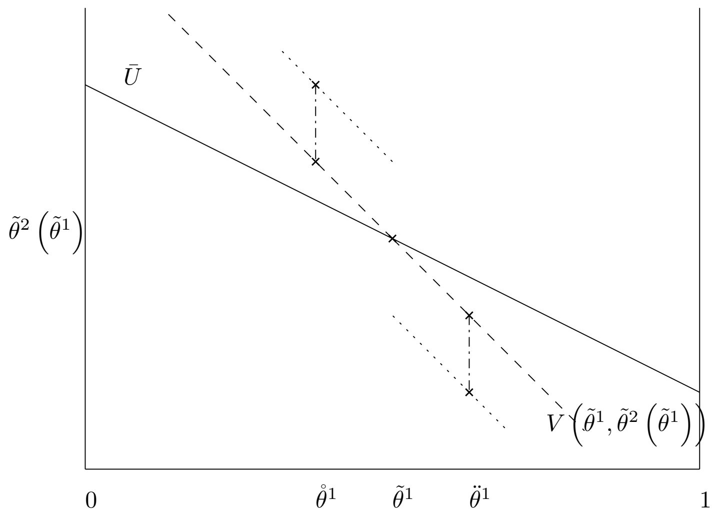


Figure 5: Second-best analysis

The above argument is illustrated in Figure 5. The area above (below) the solid line is where a sale is efficient (inefficient). If $\tau\left(\theta^1\right) = \tau\left(\tilde{\theta}^1\right)$ for every θ^1, then a sale occurs for every signal in the area above the dashed line. Now, any deviation from this pricing policy results in a loss of welfare. If for types $\mathring{\theta}^1 < \tilde{\theta}^1$ we have $\tau\left(\mathring{\theta}^1\right) > \tau\left(\tilde{\theta}^1\right)$, then a sale occurs for every signal in the area above the dotted line, and the intersection of the area where the sale is efficient and the area where the sale is executed decreases. If for types $\tilde{\theta}^1 < \ddot{\theta}^1$ we have $\tau\left(\tilde{\theta}^1\right) > \tau\left(\ddot{\theta}^1\right)$, then the intersection of the area where the sale is inefficient and the area where the sale is executed increases.

We proceed to the case where the buyer's valuation is not monotonic along the boundary of the efficient decision rule. We present sufficient conditions for the second-best solution to provide a higher expected social welfare in a sequential environment than in a static environment. The improvement upon the static mechanism is achieved through construction of a decision rule whose boundary differs from the boundary of the static second-best decision rule in a way that provides a welfare-improving allocation while maintaining sequential implementability.

Consider the second-best decision rule that is implementable in a static mechanism. We denote it by $q^{SB}(\theta)$. This decision rule takes the form of

$$q^{SB}(\theta) = \begin{cases} 1 \text{ if} & V(\theta^1, \theta^2) > C^{SB} \\ 0 \text{ or } 1 \text{ if} & V(\theta^1, \theta^2) = C^{SB} \\ 0 & \text{otherwise} \end{cases}$$

We denote by $\bar{V}^{SB}$ the boundary of the second-best static decision rule $q^{SB}(\theta)$, i.e.,

$$\bar{V}^{SB} := \left\{ \left(\theta^1, \theta^2\right) \text{ s.t. } V\left(\theta^1, \theta^2\right) = C^{SB} \right\}$$

We note that the boundary of the second-best static decision rule $\bar{V}^{SB}$ and the boundary of the efficient decision rule $\bar{U}$ intersect.[24] We denote by $\dot{\theta}^1$ the rightmost point at which these boundaries intersect, i.e.,

$$\dot{\theta}^1 := \max \left\{ \theta^1 \text{ s.t. } \left(\theta^1, \tilde{\theta}^2\left(\theta^1\right)\right) \in \bar{V}^{SB} \cap \bar{U} \right\}$$

We denote by $\ddot{\theta}^1$ the leftmost point at which these boundaries intersect, i.e.,

$$\ddot{\theta}^1 := \min \left\{ \theta^1 \text{ s.t. } \left(\theta^1, \tilde{\theta}^2\left(\theta^1\right)\right) \in \bar{V}^{SB} \cap \bar{U} \right\}$$

We now present sufficient conditions for improving the second-best solution by a sequential mechanism.

Theorem 8. *Assume one of the following conditions holds: (1) for every $\theta^1 > \dot{\theta}^1$ we have that $V\left(\dot{\theta}^1, \tilde{\theta}^2\left(\dot{\theta}^1\right)\right) > V\left(\theta^1, \tilde{\theta}^2(\theta^1)\right)$ or (2) for every $\theta^1 < \ddot{\theta}^1$ we have that $V\left(\theta^1, \tilde{\theta}^2(\theta^1)\right) > V\left(\ddot{\theta}^1, \tilde{\theta}^2\left(\ddot{\theta}^1\right)\right)$. Then there exists a decision rule that is sequentially implementable and provides a higher expected welfare than $q^{SB}(\theta)$.*

The idea of the proof is as follows. Assume for example that (1) holds. This means that at any point that is to the right of $\dot{\theta}^1$, the boundary of the second-best static decision rule lies above the boundary of the efficient decision rule. Therefore, we can

[24] In Appendix A we characterize the second-best mechanism in a static environment and show that this property holds.

construct a decision rule $\tilde{q}(\theta)$ with two properties. The first is that to the left of $\dot{\theta}^1$ the boundary of the decision rule $\tilde{q}(\theta)$ coincides with the boundary of $q^{SB}(\theta)$, while to the right of $\dot{\theta}^1$ the boundary of the decision rule $\tilde{q}(\theta)$ is below the boundary of $q^{SB}(\theta)$ and above the boundary of the efficient decision rule. This property implies that $\tilde{q}(\theta)$ provides a higher expected welfare than $q^{SB}(\theta)$. The second property is that the buyer's valuation is decreasing as we move rightward along the boundary of $\tilde{q}(\theta)$. This property implies that $\tilde{q}(\theta)$ is sequentially implementable. Such a construction is illustrated in Figure 6.

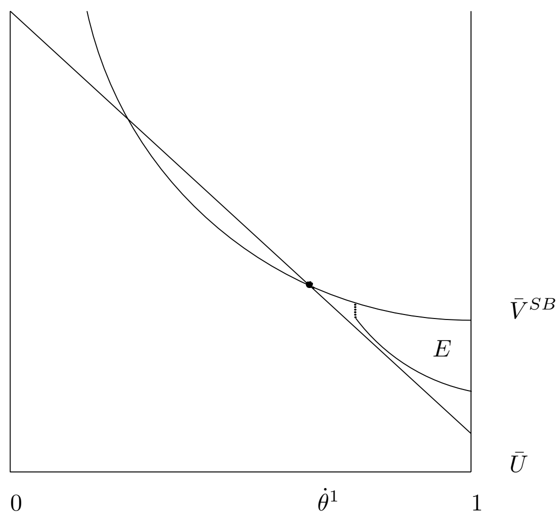


Figure 6: The set where $q^{SB}(\theta)$ and $\tilde{q}(\theta)$ do not coincide is denoted by E.

3 Multiple Buyers

In this section we generalize the results of the previous section to the case of multiple buyers. We consider a seller whose valuation for the item is zero and whose objective is to attain efficiency. In such a scenario the seller will want to sell the item to the buyer who values it the most. As in the single-buyer case, the possibility of static efficient implementation depends on the dimensionality of a buyer's information and on whether or not this information has externalities on the valuations of other buyers. In situations where buyers' valuations are independent, the seller can implement an efficient sale in dominant strategies by executing the celebrated VCG mechanism.[25] When the buyers' valuations are interdependent and each buyer's information is unidimensional efficiency is attainable if each buyer's valuation satisfies a single-crossing condition.[26] However, when buyers' valuations are interdependent and their information is independent and multidimensional, efficient implementation is generically impossible, as Maskin (1992), Dasgupta and Maskin (2000), and Jehiel and Moldovanu (2001) show. In particular their result holds in the setting of linear valuations. Our first result shows that in this setting efficiency is not generically impossible in sequential environments. That is, efficiency can be implementable on a positive measure set of valuations in sequential environments.

Bikhchandani (2006) focuses on a general set of valuations that satisfy a certain single-crossing condition. Bikhchandani considers static mechanisms that satisfy conditional efficiency, i.e., mechanisms that whenever they allocate the item it is to the buyer with the highest valuation. Bikhchandani constructs the most efficient mechanism in the set of ex-post implementable mechanisms that satisfy conditional efficiency.[27] This mechanism is typically inefficient since the seller must keep the item on a set of signals of a positive measure and gains from trade are not realized.[28] Our second result shows that it is possible to implement a more efficient allocation in sequential environments while maintaining conditional efficiency. That is, we show that there exists an ex-post implementable, conditionally efficient sequential mechanism that is more efficient than Bikhchandani's mechanism.

3.1 Setup

Consider a seller of a single item facing a set I of n potential buyers, $I = \{1, ..., n\}$. There are two periods, 1 and 2. Each buyer $i \in I$ receives a private signal $\theta_i^1 \in [0, 1]$ in period 1, and a private signal $\theta_i^2 \in [0, 1]$ in period 2. We let $\theta_i := (\theta_i^1, \theta_i^2)$ and denote

[25] See, Vickrey (1961), Clarke (1971), and Groves (1973).

[26] See, for example, Maskin (1992), Dasgupta and Maskin (2000), and Perry and Reny (2002).

[27] Jehiel et al. (2006) show that in environments of interdependent valuations and multidimensional signals ex-post implementation of non-trivial deterministic decision rules is generically impossible. Bikhchandani (2006) shows that in private good settings, ex-post implementation of non-trivial deterministic decision rules is possible on a non-negligible set of valuations.

[28] Moreover, this mechanism need not be efficient under incentive compatibility constraints.

by θ_{-i} the signals of buyers other than i. The signal θ_i^2 is independent of θ_i^1 and θ_{-i} and is uniformly distributed, and this is common knowledge. Each buyer's payoff is minus his payment to the seller, plus, in case he gets the item, the value of the item. Buyer i's valuation of the item is a function of the agents' signals, $V_i : \Theta \to \mathbb{R}_+$, where $\Theta := [0,1]^{2n}$. We assume that for every $i \in I$ his valuation V_i is continuously differentiable and strictly increasing in θ_i^1 and θ_i^2 for every θ_{-i}. We denote by A the set of feasible allocations $A = \{a_i\}_{i \in I} \cup \{a_0\}$, where a_i is the allocation where agent i receives the item, and a_0 is the allocation where the seller keeps the object. A *decision rule* is a function $q : \Theta \to A$. Given a decision rule $q(\theta)$ we define for every $i \in I$ the following function:

$$q_i(\theta) = \begin{cases} 1 \text{ if} & q(\theta) = a_i \\ 0 & \text{otherwise} \end{cases}$$

A *social choice function, s*, assigns an allocation and payment scheme to every realization of signals, i.e., $s(\theta) = (q(\theta), t_1(\theta), ..., t_n(\theta))$, where $q(\theta) \in A$ and $t_i(\theta) \in \mathbb{R}$ for every $i \in I$. We say that a social choice function $(q(\theta), t_1(\theta), ..., t_n(\theta))$ is *ex-post implementable in a sequential mechanism* if for every $i \in I$ and every θ_{-i} we have

1.
$$E_{\theta_i^2}\left[V_i\left(\theta_i^1, \theta_i^2, \theta_{-i}\right) \cdot q_i\left(\theta_i^1, \theta_i^2, \theta_{-i}\right) - t_i\left(\theta_i^1, \theta_i^2, \theta_{-i}\right)\right] \geq$$
$$E_{\theta_i^2}\left[V_i\left(\theta_i^1, \theta_i^2, \theta_{-i}\right) \cdot q_i\left(\hat{\theta}_i^1, \hat{\theta}_i^2\left(\theta_i^2\right), \theta_{-i}\right) - t_i\left(\hat{\theta}_i^1, \hat{\theta}_i^2\left(\theta_i^2\right), \theta_{-i}\right)\right]$$
for every $\theta_i^1 \in [0,1]$ and $\hat{\theta}_i^1 \in [0,1]$ and every $\hat{\theta}_i^2 : [0,1] \to [0,1]$.

2.
$$V_i\left(\theta_i^1, \theta_i^2, \theta_{-i}\right) \cdot q_i\left(\theta_i^1, \theta_i^2, \theta_{-i}\right) - t_i\left(\theta_i^1, \theta_i^2, \theta_{-i}\right) \geq$$
$$V_i\left(\theta_i^1, \theta_i^2, \theta_{-i}\right) \cdot q_i\left(\theta_i^1, \hat{\theta}_i^2, \theta_{-i}\right) - t_i\left(\theta_i^1, \hat{\theta}_i^2, \theta_{-i}\right)$$
for every $\hat{\theta}_i^2 \in [0,1]$.

The above solution concept is robust in the sense that even if the buyer knows the realization of the other buyers' signals he will still choose to report truthfully given that the other buyers report truthfully. In Section 2.3.1 we showed that, by an appropriate modification of the assumption on the buyer's valuation function, our analysis can be applied to any distribution of the buyer's signals. The same argument is valid in the case of multiple buyers when we consider the conditional distribution $F_i\left(\theta_i^2|\theta_i^1,\theta_{-i}\right)$ for each agent i. The notion of sequential ex-post implementation therefore requires that the functions $V_i\left(\theta_i^1,\theta_i^2,\theta_{-i}\right)$ and the marginal probabilities $F_i\left(\theta_i^2|\theta_i^1,\theta_{-i}\right)$ are commonly known. Nonetheless, a buyer does not need to know the distribution of other buyers' signals. In that sense sequential ex-post implementation is more robust than Bayesian implementation, which requires that the latter requirement holds. However, it is less robust than static ex-post implementation, which does not require any assumption on knowledge of the properties of the joint distribution $F(\theta)$.

From the above definition we can deduce that the set of ex-post implementable decision rules in the case of multiple buyers is characterized as the set of decision rules for which the necessary and sufficient conditions for implementation in the single-buyer case apply to every buyer for any realization of other buyers' signals. In particular, we can deduce the following corollaries:

Corollary 9. *A decision rule $q(\theta)$ is ex-post implementable in a static mechanism if and only if for every $i \in I$ and every θ_{-i} there exists a number $C_i(\theta_{-i}) \in \mathbb{R}$ such that*

$$q_i(\theta) = \begin{cases} 1 \text{ if} & V_i\left(\theta_i^1,\theta_i^2,\theta_{-i}\right) > C_i(\theta_{-i}) \\ 0 \text{ or } 1 \text{ if} & V_i\left(\theta_i^1,\theta_i^2,\theta_{-i}\right) = C_i(\theta_{-i}) \\ 0 & \text{otherwise} \end{cases}$$

Corollary 10. *A decision rule $q(\theta)$ is ex-post implementable in a sequential mechanism if and only if for every $i \in I$ and every θ_{-i} there exists a function $C_i(\cdot,\theta_{-i}) \in \mathtt{C}$ such that*

$$q_i(\theta) = \begin{cases} 1 \text{ if} & \theta_i^2 > C_i\left(\theta_i^1,\theta_{-i}\right) \\ 0 \text{ or } 1 \text{ if} & \theta_i^2 = C_i\left(\theta_i^1,\theta_{-i}\right) \\ 0 & \text{otherwise} \end{cases}$$

and in addition the function $V_i\left(\theta_i^1, C_i\left(\theta_i^1,\theta_{-i}\right),\theta_{-i}\right)$ is a decreasing function of θ_i^1 in the segment $\left[\underline{\theta}_i^{1,C_i}(\theta_{-i}), \overline{\theta}_i^{1,C_i}(\theta_{-i})\right]$ for every θ_{-i}.

Corollary 11. *Consider a decision rule of the following form: for every $i \in I$,*

$$q_i(\theta) = \begin{cases} 1 \text{ if} & U_i(\theta_i^1, \theta_i^2, \theta_{-i}) > C_i(\theta_{-i}) \\ 0 \text{ or } 1 \text{ if} & U_i(\theta_i^1, \theta_i^2, \theta_{-i}) = C_i(\theta_{-i}) \\ 0 & \text{otherwise} \end{cases}$$

where $U_i(\theta_i^1, \theta_i^2, \theta_{-i})$ is continuously differentiable and strictly increasing in θ_i^1 and θ_i^2. The decision rule $q(\theta)$ is implementable by a sequential mechanism if and only if for every $i \in I$ and for every θ_{-i} we have

$$\frac{\partial V_i / \partial \theta_i^1}{\partial V_i / \partial \theta_i^2}\left(\theta_i^1, \theta_i^2, \theta_{-i}\right) \leq \frac{\partial U_i / \partial \theta_i^1}{\partial U_i / \partial \theta_i^2}\left(\theta_i^1, \theta_i^2, \theta_{-i}\right)$$

for every $(\theta_i^1, \theta_i^2) \in \bar{U}_i(\theta_{-i}) := \{(\theta_i^1, \theta_i^2) \text{ s.t. } U_i(\theta_i^1, \theta_i^2, \theta_{-i}) = C_i(\theta_{-i})\}$

Note that in order for the decision rule $q(\theta)$ to be well defined it must be *admissible*; that is, it should allocate the item to at most one buyer. This means that if $q_i(\theta) = 1$ then $q_j(\theta) = 0$ for every $j \in I \setminus \{i\}$. Therefore, $q(\theta)$ is implementable if and only if it is admissible, and for every i the function $q_i(\theta)$ satisfies the conditions of the above corollaries.

3.2 Efficient Implementation in Sequential Environments

In this subsection we present the first result of the multiple-buyers model. We show that when the buyers' valuation functions are linear then the generic impossibility result of Jehiel and Moldovanu (2001) regarding Bayesian efficient implementation in static settings does not extend to sequential environments. To illustrate this observation we consider a setup where there are two potential buyers, 1 and 2, and the buyers' valuations are

$$V_1(\theta) = \alpha_1^1 \cdot \theta_1^1 + \alpha_1^2 \cdot \theta_1^2 + \beta_1^1 \cdot \theta_2^1 + \beta_1^2 \cdot \theta_2^2$$

$$V_2(\theta) = \alpha_2^1 \cdot \theta_1^1 + \alpha_2^2 \cdot \theta_1^2 + \beta_2^1 \cdot \theta_2^1 + \beta_2^2 \cdot \theta_2^2$$

where

$$\alpha_1^k > \alpha_2^k > 0 \text{ and } \ \beta_2^k > \beta_1^k > 0 \ \text{ for } k \in \{1,2\}$$

We define $U_i(\theta) := V_i(\theta) - V_j(\theta)$ for every $i \in \{1,2\}$. The efficient decision rule is admissible and it takes the following form:

$$q_i(\theta) = \begin{cases} 1 \text{ if} & U_i(\theta) > 0 \\ 0 \text{ or } 1 \text{ if} & U_i(\theta) = 0 \\ 0 & \text{otherwise} \end{cases}$$

for every $i \in \{1,2\}$. In static settings, Jehiel and Moldovanu (2001) show that if the buyers' signals are independent then the efficient decision rule is Bayesian implementable only if

$$\frac{\alpha_1^1}{\alpha_1^2} = \frac{\alpha_2^1}{\alpha_2^2} \quad \text{and} \quad \frac{\beta_2^1}{\beta_2^2} = \frac{\beta_1^1}{\beta_1^2}$$

These conditions are met on a set of measure zero in $\mathbb{R}_+^8$; that is, efficiency is generically impossible. However, Corollary 11 implies that in the sequential environment the efficient decision rule is ex-post implementable by a sequential mechanism if

$$\frac{\alpha_1^1}{\alpha_1^2} \geq \frac{\alpha_2^1}{\alpha_2^2} \quad \text{and} \quad \frac{\beta_2^1}{\beta_2^2} \geq \frac{\beta_1^1}{\beta_1^2}$$

These conditions are met on a set of positive measure in $\mathbb{R}_+^8$; that is, efficiency is attained on a non-negligible set of valuations.

3.3 Conditional Efficiency in Sequential Environments

As mentioned earlier, Bikhchandani (2006) considers environments where buyers' valuations satisfy a single-crossing condition, i.e., where a buyer's valuation is more responsive to his signal than other buyers' valuations. Bikhchandani proposes a mechanism that satisfies conditional efficiency (henceforth CE) and is the most efficient mechanism in the set of ex-post implementable CE static mechanisms. In this subsection we show that in a sequential environment, for a general set of valuations, there exists an ex-post implementable sequential mechanism that is CE and is more efficient than Bikhchandani's mechanism. The improvement upon Bikhchandani's mechanism is achieved through construction of a decision rule whose boundary differs from the boundary of Bikhchandani's decision rule in a way that provides a more efficient allocation while maintaining sequential implementability.

Remark. In general environments of interdependent valuations and multidimensional signals, Jehiel et al. (2006) show that for generic valuation functions the only deterministic decision rules that are ex-post implementable are trivial. Their impossibility result depends on the assumption that for any two alternatives there exist at least two agents who are not indifferent between them. The possibility of implementing non-trivial decision rules in private-goods settings arises since all the buyers except one are indifferent between the alternative where this non-indifferent buyer wins the item and the alternative where the seller keeps the item.

To illustrate our result we consider the following setup. There are two buyers $I = \{1, 2\}$ and each buyer $i \in I$ receives a two-dimensional signal (θ_i^1, θ_i^2). Bikhchandani's decision rule takes the following form. For every $i \in I$,

$$q_i^B(\theta) = \begin{cases} 1 \text{ if} & V_i(\theta_i^1, \theta_i^2, \theta_j) > C_i^B(\theta_j) \\ 0 \text{ or } 1 \text{ if} & V_i(\theta_i^1, \theta_i^2, \theta_j) = C_i^B(\theta_j) \\ 0 & \text{otherwise} \end{cases}$$

Consider an arbitrary θ_j. We denote by $\bar{V}_i^B(\theta_j)$ the boundary of the decision rule $q_i^B(\theta)$ given θ_j, i.e.,

$$\bar{V}_i^B(\theta_j) := \left\{ \left(\theta_i^1, \theta_i^2\right) \text{ s.t. } V_i\left(\theta_i^1, \theta_i^2, \theta_j\right) = C_i^B(\theta_j) \right\}$$

Since Bikhchandani's decision rule is CE the signals of the decision rule's boundary $\bar{V}_i^B(\theta_j)$ lie above the signals of the boundary of the efficient decision rule $\bar{U}_i(\theta_j)$. Bikhchandani's decision rule also has the property that the sets $\bar{V}_i^B(\theta_j)$ and $\bar{U}_i(\theta_j)$ intersect. We denote by $\dot{\theta}_i^1(\theta_j)$ the rightmost point where $\bar{V}_i^B(\theta_j)$ and $\bar{U}_i(\theta_j)$ intersect, i.e.,[29]

$$\dot{\theta}_i^1(\theta_j) := \max\left\{ \theta_i^1 \text{ s.t. } \left(\theta_i^1, \tilde{\theta}_i^2\left(\theta_i^1, \theta_j\right)\right) \in \bar{V}_i^B(\theta_j) \cap \bar{U}_i(\theta_j) \right\}$$

[29]Where $\tilde{\theta}_i^2\left(\theta_i^1, \theta_j\right) := \theta_i^2$ s.t. $\left(\theta_i^1, \theta_i^2\right) \in \bar{U}_i(\theta_j)$.

We denote by $[\underline{u}_i(\theta_j), \overline{u}_i(\theta_j)]$ the segment of all θ_i^1 such that there exist θ_i^2 for which $(\theta_i^1, \theta_i^2) \in \bar{U}_i(\theta_j)$. Now, assume that $\dot{\theta}_i^1(\theta_j) < \overline{u}_i(\theta_j)$; then for every point to the right of $\dot{\theta}_i^1(\theta_j)$ we have that boundary $\bar{V}_i^B(\theta_j)$ lies above the boundary of the efficient decision rule $\bar{U}_i(\theta_j)$. Therefore, we can apply a similar construction to the one that appears in the proof of Theorem 8. That is, we can construct a function $\tilde{q}_i(\theta_i, \theta_j)$ that satisfies the condition of Corollary 10. In addition, to the left of $\dot{\theta}^1$ the boundary of $\tilde{q}_i(\theta_i, \theta_j)$ coincides with the boundary of $q_i^B(\theta_i, \theta_j)$ and to the right of $\dot{\theta}^1$ the boundary of $\tilde{q}_i(\theta_i, \theta_j)$ is below the boundary of $q_i^B(\theta_i, \theta_j)$ and above the boundary of the efficient decision rule. This means that $\tilde{q}_i(\theta_i, \theta_j)$ and $q_i^B(\theta_i, \theta_j)$ coincide except in a set of signals where $\tilde{q}_i(\theta_i, \theta_j)$ assigns the item to agent i while $q_i^B(\theta_i, \theta_j)$ assigns the item to the seller, who receives no utility from the item. Therefore, $\tilde{q}_i(\theta_i, \theta_j)$ provides a greater expected social welfare than $q_i^B(\theta_i, \theta_j)$.

Now, if we have that $\dot{\theta}_i^1(\theta_j) < \overline{u}_i(\theta_j)$ on a positive measure of Θ_j, we can construct a decision rule by changing Bikhchandani's decision rule from $q_i^B(\theta_i, \theta_j)$ to $\tilde{q}_i(\theta_i, \theta_j)$ in this subset of Θ_j. This new decision rule is CE (and hence admissible), is ex-post implementable in a sequential mechanism, and is strictly more efficient than Bikhchandani's decision rule.

3.4 Remark on Constrained Efficiency

We showed that in a subclass of mechanisms that satisfy conditional efficiency, it is generally possible to achieve welfare improvement by using sequential mechanisms. However, as Bikhchandani points out, in a static environment the most efficient conditionally efficient mechanism need not be constrained efficient.[30] A question that arises is whether we can present a general characterization of settings in which a sequential mechanism can provide a higher expected social welfare than a constrained efficient static mechanism. In this subsection we explain the complexity of finding such a characterization for general environments that do not make very particular assumptions on the valuation functions.

Consider an agent i and a profile of signals of other agents θ_{-i}. We say that a

[30] A mechanism is constrained efficient if it supplies the highest expected welfare out of the mechanisms that are implementable.

function $\tilde{q}_i(\theta_i, \theta_{-i})$ (strictly) *dominates* the function $q_i(\theta_i, \theta_{-i})$ if the set of signals of agent i for which $q_i(\theta_i, \theta_{-i}) = 1$ is (strictly) contained within the set of signals where $\tilde{q}_i(\theta_i, \theta_{-i}) = 1$, and if the set of signals where these sets do not coincide is contained in the set of signals where it is efficient to allocate the item to agent i. A decision rule $\tilde{q}$ *strictly dominates* the decision rule q if for every $i \in I$ and every θ_{-i} we have that $\tilde{q}_i(\theta_i, \theta_{-i})$ dominates $q_i(\theta_i, \theta_{-i})$ and there is at least one agent k for which $\tilde{q}_k(\theta_k, \theta_{-k})$ strictly dominates $q_k(\theta_k, \theta_{-k})$ on a set of positive measure of θ_{-k}. For general environments of valuation functions the way to show that a decision rule provides a higher expected welfare than another decision rule is to show that the former strictly dominates the latter. Theorem 8 and the result in Section 3.3 are both achieved by construction of a sequentially implementable decision rule that strictly dominates the second-best static decision rule. However, in general environments of multiple agents with interdependent valuations and multidimensional signals it is not straightforward to construct a sequentially implementable decision rule that strictly dominates the constrained efficient decision rule. We illustrate this point below.

Consider the two-buyer setup of Section 3.3. We denote by $q^{SB}(\theta)$ the constrained efficient decision rule. Fix some $\tilde{\theta}_j$. Denote by $\bar{V}_i^{SB}\left(\tilde{\theta}_j\right)$ the boundary of $q_i^{SB}\left(\theta_i, \tilde{\theta}_j\right)$, and denote by $\dot{\theta}_i^1\left(\tilde{\theta}_j\right)$ the rightmost point where $\bar{V}_i^{SB}\left(\tilde{\theta}_j\right)$ and $\bar{U}_i\left(\tilde{\theta}_j\right)$ intersect. Assume that to the right of $\dot{\theta}_i^1\left(\tilde{\theta}_j\right)$ the boundary of the second-best decision rule $\bar{V}_i^{SB}\left(\tilde{\theta}_j\right)$ lies above the boundary of the efficient decision rule $\bar{U}_i\left(\tilde{\theta}_j\right)$. Now assume that we apply a similar construction to the one we applied in the proof of Theorem 8. That is, we construct a new decision rule $\tilde{q}(\theta)$ in which to the left of $\dot{\theta}^1$ the boundary of $\tilde{q}_i\left(\theta_i, \tilde{\theta}_j\right)$ coincides with the boundary of $q_i^{SB}\left(\theta_i, \tilde{\theta}_j\right)$ and to the right of $\dot{\theta}^1$ the boundary of $\tilde{q}_i\left(\theta_i, \tilde{\theta}_j\right)$ is below the boundary of $q_i^{SB}\left(\theta_i, \tilde{\theta}_j\right)$ and above the boundary of the efficient decision rule $\bar{U}_i(\theta_j)$. Unless we can preclude that in the set of signals for which $\tilde{q}_i\left(\theta_i, \tilde{\theta}_j\right) = 1$ and $q_i^{SB}\left(\theta_i, \tilde{\theta}_j\right) = 0$ there are no signals that allocate the object to agent j under the decision rule $q^{SB}(\theta)$, we run into the following problem. Assume that such a signal exists and denote it by $\left(\tilde{\theta}_i, \tilde{\theta}_j\right)$. Under $q^{SB}(\theta)$ we have that $q_j^{SB}\left(\tilde{\theta}_j, \tilde{\theta}_i\right) = 1$. On the other hand, admissibility implies that under $\tilde{q}(\theta)$ we have that $\tilde{q}_j\left(\tilde{\theta}_j, \tilde{\theta}_i\right) = 0$. Fix $\tilde{\theta}_i$. Since $q_j^{SB}\left(\tilde{\theta}_j, \tilde{\theta}_i\right) = 1$ and $\tilde{q}_j\left(\tilde{\theta}_j, \tilde{\theta}_i\right) = 0$, we get that the boundary of $\tilde{q}_j\left(\theta_j, \tilde{\theta}_i\right)$ differs from the boundary of $q_j^{SB}\left(\theta_j, \tilde{\theta}_i\right)$ at one point. Moreover, the incentive constraints of buyer j imply further differences in these boundaries. That is, we get that under the new decision rule $\tilde{q}(\theta)$ we have that the boundary of $\tilde{q}_j\left(\theta_j, \tilde{\theta}_i\right)$ differs from the boundary of $q_j^{SB}\left(\theta_j, \tilde{\theta}_i\right)$. However, it is not clear that this change in the boundaries satisfies that $\tilde{q}_j\left(\theta_j, \tilde{\theta}_i\right)$ dominates $q_j^{SB}\left(\theta_j, \tilde{\theta}_i\right)$. Hence it is not clear that $\tilde{q}(\theta)$ strictly dominates $q^{SB}(\theta)$.

A way to construct a decision rule $\tilde{q}(\theta)$ that dominates the constrained efficient decision rule $q^{SB}(\theta)$ is therefore to find an agent i where there is a positive measure set $\tilde{\Theta}_j \subseteq \Theta_j$ such that for every $\tilde{\theta}_j \in \tilde{\Theta}_j$ we have that $\tilde{q}_i\left(\theta_i, \tilde{\theta}_j\right)$ strictly dominates the function $q_i^{SB}\left(\theta_i, \tilde{\theta}_j\right)$ and for every signal θ_i for which $\tilde{q}_i\left(\theta_i, \tilde{\theta}_j\right) = 1$ and $q_i^{SB}\left(\theta_i, \tilde{\theta}_j\right) = 0$ we have that $q_j^{SB}\left(\theta_i, \tilde{\theta}_j\right) = 0$ (i.e., the seller keeps the object in this signal under $q^{SB}(\theta)$). The characterization of the sets of valuations in which these properties hold is left for future work.

4 Concluding Remarks

We have considered the problem of implementing an efficient sale in an environment where buyers' signals are multidimensional and there exist informational externalities. In these environments efficiency is typically unattainable if signals arrive all at once. We have shown that if signals arrive sequentially then it is possible to attain efficiency in such environments. Moreover, even if sequential mechanisms do not attain full efficiency they can still provide more efficient second-best solutions than static mechanisms. We see a number of directions for future research, such as finding efficient sequential mechanisms that are detail-free, characterizing the conditions that enable full efficiency to be achieved in environments where the dimension of the buyer's signals is greater than two and/or where seller has multiple objects and/or where there are allocative externalities.

Appendix

A Generality of the Results

In the present paper we have restricted our attention to direct deterministic mechanisms. In this appendix we analyze for which results of this paper this restriction is without loss of generality. We consider two generalizations. The first is to the set of indirect deterministic mechanisms. As Strausz (2003) points out, even if one restricts attention to deterministic mechanisms, it is not clear that the restriction to direct deterministic mechanisms is without loss of generality. This is because agents may play mixed strategies in the indirect deterministic mechanism and therefore the direct mechanism that mimic the equilibrium strategies of the indirect mechanism is not deterministic. We find that all the results of the paper still hold even if we consider indirect deterministic mechanisms. This outcome is based on the observation that for every implementable indirect deterministic mechanism there exists an implementable direct deterministic mechanism that yields equal or greater social welfare. The second generalization is to stochastic mechanisms. Such a generalization is less crucial than the first generalization because the restriction to deterministic mechanisms can be justified by practical considerations that derive from the commitment assumption. The commitment assumption suggests that the outcomes of a mechanism can be executed in an enforceable manner. Stochastic mechanisms are problematic in this respect since they require that the seller hold a randomization device and that the results of this device be verifiable. Laffont and Martimort (2002) note that

> "Ensuring this verifiability is a more difficult problem than ensuring that a deterministic mechanism is enforced, because any deviation away from a given randomization can only be statistically detected once sufficiently many realizations of the contracts have been observed. [...] The enforcement of such stochastic mechanisms is thus particularly problematic."

We find that all the results of the paper except maybe for Theorem 7 can be generalized to environments with stochastic mechanisms.

The results that appear in Section 3 have to do with existence of mechanisms. Therefore, restriction to a subset of mechanisms does not undermine them. We proceed to show that our results can be generalized to the set of indirect deterministic mechanisms. We show that for every implementable indirect deterministic mechanism there exists an implementable direct deterministic mechanism that yields equal or greater social welfare. An implementable deterministic static mechanism yields two alternatives to the buyer (buy the item or don't buy the item) and assigns to each such alternative a single price. Therefore, the only set of signals where the buyer can be indifferent between the two alternatives (and play mixed strategies) are on the isovalue curve where the buyer's valuation equals the difference in transfers. This set is of measure zero and does not affect the expected social welfare. Hence, a direct mechanism that arbitrarily assigns to the signals in this set a single alternative yields the same expected welfare. An implementable sequential deterministic mechanism yields two alternatives in the second period (buy the item or don't buy the item) and assigns to each such alternative a single price. Therefore, there is a single signal in the second period where the buyer can be indifferent between the two alternatives. Of course, allowing for mixed strategies at this point would not change the expected social welfare. In the first period of an implementable sequential deterministic mechanism every type of the buyer may randomize between several options that this type is indifferent among. Each option is composed of a single price in the second period and a payment today. Denote by $I(\theta^1)$ the support of options that type θ^1 is mixing. Implementability implies that every $a \in I(\theta^1)$ is preferred by type θ^1 to every $b \in I\left(\tilde{\theta}^1\right)$ and every $b \in I\left(\tilde{\theta}^1\right)$ is preferred by type $\tilde{\theta}^1$ to every $a \in I(\theta^1)$. Therefore, every mechanism that offers some arbitrary a in $I(\theta^1)$ to type θ^1 is implementable by a direct mechanism. Now, each second-period price sets an expected social welfare given θ^1. Consider the option $a^*(\theta^1) \in I(\theta^1)$ that sets the second-period price that maximizes this expected social welfare given θ^1 out of all the options in[31] $I(\theta^1)$. The deterministic mechanism for which $a^*(\theta^1) = I(\theta^1)$ yields

[31]Such an option exists because the support of the second-period prices in $I\left(\theta^1\right)$ is a closed set. Denote by $I_h\left(\theta^1\right)$ the set of options in $I\left(\theta^1\right)$ that set a price that is greater than or equal to the price that is set by the efficient decision rule. Denote by $I_l\left(\theta^1\right)$ the set of options in $I\left(\theta^1\right)$ that set a price that is less than or equal to the price that is set by the efficient decision rule. We have that $a^*\left(\theta^1\right)$ is either the option that sets the minimum price in $I_h\left(\theta^1\right)$ or the option that sets the maximal price in $I_l\left(\theta^1\right)$.

equal or greater expected welfare than the original mechanism and is implementable by a direct mechanism.

We now consider the generalization of our results to stochastic mechanism. Since we assume that the buyer's utility is linear in transfer we can assume without loss of generality that the transfers are deterministic. Therefore, a deterministic decision rule is implementable by a stochastic mechanism if and only if it is implementable by a deterministic mechanism. Since efficient decision rules are (almost everywhere) deterministic, all the result about the possibility of implementing full efficiency are without loss of generality. We now show that the second-best decision rule in a static environment is also deterministic. This implies that Theorem 8 is also without loss of generality. Denote the valuation of the seller if she keeps the item by V_s and the valuation of the buyer if he gets the item by V_b. We assume that the following condition holds: $V_b(\theta') - V_b(\theta) > V_s(\theta') - V_s(\theta)$ for every $\theta' > \theta$ where[32] $\theta', \theta \in [0,1]^2$ .Consider the buyer's isovalue curves in $[0,1]^2$ and let

$$V_I(V) = \left\{\theta \in [0,1]^2 \text{ s.t } V_b(\theta) = V\right\}$$

We define the following function:

$$W(V) := E_{\theta \in V_I(V)}\left[V_b(\theta) - V_s(\theta)\right]$$

This function is strictly increasing in V. Now consider an equilibrium that is *regular*, i.e., where every θ and $\tilde{\theta}$ that are on the same isovalue curve of the buyer map the same (possibly stochastic) outcome.[33] We denote by V^* the value for which[34] $W(V^*) = 0$. In that case we get that the second-best decision rule out of the set of all stochastic

[32]Let $x' > x$ denote that x' is at least as large as x in every coordinate and $x' \neq x$.
[33]The term "regular equilibrium" is due to Dasgupta and Maskin (2000).
[34]If no such value exists then the efficient decision rule is trivial and implementable.

decision rules is

$$q(\theta) = \begin{cases} 1 & \text{if } \theta \in V_I(V) \text{ s.t. } V > V^* \\ r \in [0,1] & \text{if } \theta \in V_I(V^*) \\ 0 & \text{if } \theta \in V_I(V) \text{ s.t. } V < V^* \end{cases}$$

where r is the probability that the item is assigned to the buyer. That is, the second best decision rule is also (almost everywhere) deterministic.

B Proofs

Proof of Claim 1

In the main text we showed that there can be at most one isovalue curve for which two signals that lie on this isovalue curve are assigned different alternatives. This means that the decision rule maps according to values of V. Assume that there exists a valuation $V(\theta)$ such that $q(\theta) = 1$ and a valuation $V(\theta')$ such that $q(\theta') = 0$ and $V(\theta') > V(\theta)$. Implementability implies that

$$V(\theta) - t(1) \geq -t(0)$$

and

$$V(\theta') - t(1) \leq -t(0)$$

and so we get $V(\theta) \geq V(\theta')$, a contradiction. This proves necessity. We now prove sufficiency. We set $t(0) = 0$ and $t(1) = C$; then we get that for every θ such that $V(\theta) < C$ we have $V(\theta) - t(1) < 0$ and for every θ such that $V(\theta) \geq C$ we have $V(\theta) - t(1) \geq 0$. Thus, the decision rule is implementable by a static mechanism ■

Proof of Theorem 2

Lemma A: Condition 2 in the definition of *implementation in a sequential mechanism* is satisfied iff for every θ^1 there exists $C(\theta^1)$ such that

$$q(\theta) = \begin{cases} 1 \text{ if} & \theta^2 > C(\theta^1) \\ 0 \text{ or } 1 \text{ if} & \theta^2 = C(\theta^1) \\ 0 & \text{otherwise} \end{cases}$$

and the transfers $t(q(\theta), \theta^1) + p(\theta^1)$ are set as follows: $t(1, \theta^1) = V(\theta^1, C(\theta^1)) + p(\theta^1)$ and $t(0, \theta^1) = p(\theta^1)$

Proof : Assume that the mechanism is of the above shape; then if it is the case that either $C(\theta^1) = 1$ and $q(\theta^1, 1) = 0$ or $C(\theta^1) = 0$ and $q(\theta^1, 0) = 1$, then the mechanism in the second period is trivial and hence incentive compatible. Assume that the decision rule is not trivial. Consider some $C(\theta^1) \in [0, 1]$ then if $\theta^2 > C(\theta^1)$ and the buyer reports truthfully then the buyer gets the item and pays $t(1, \theta^1)$ and receives a utility of $V(\theta^1, \theta^2) - V(\theta^1, C(\theta^1)) - p(\theta^1)$ and if the buyer deviates he receives either the same utility or $-p(\theta^1)$. By the monotonicity of V we have that if $\theta^2 > C(\theta^1)$ then $V(\theta^1, \theta^2) - V(\theta^1, C(\theta^1)) - p(\theta^1) > -p(\theta^1)$; that is, there is no profitable deviation. If $\theta^2 < C(\theta^1)$ and the buyer reports truthfully then he does not get the item and pays $t(0, \theta^1)$ and receives a utility of $-p(\theta^1)$ and if the buyer deviates he receives either the same utility or $V(\theta^1, \theta^2) - V(\theta^1, C(\theta^1)) - p(\theta^1)$. By the monotonicity of V we have that if $\theta^2 < C(\theta^1)$ then $V(\theta^1, \theta^2) - V(\theta^1, C(\theta^1)) - p(\theta^1) \leq -p(\theta^1)$; that is, there is no profitable deviation. If $\theta^2 = C(\theta^1)$ then in either case the buyer receives a utility of $-p(\theta^1)$ and so there is no profitable deviation.

Assume that the mechanism is incentive compatible in the second period. Assume that the mechanism is not trivial. Then for every θ^2 such that $q(\theta^1, \theta^2) = 1$ we have $V(\theta^1, \theta^2) \geq t(1, \theta^1) - t(0, \theta^1)$, and for every θ^2 such that $q(\theta^1, \theta^2) = 0$ we have $V(\theta^1, \theta^2) \leq t(1, \theta^1) - t(0, \theta^1)$. Since V is continuous and monotonic there exists a single number $C(\theta^1)$ that satisfies $V(\theta^1, C(\theta^1)) = t(1, \theta^1) - t(0, \theta^1)$. Therefore, if $\theta^2 = C(\theta^1)$ then $q(\theta^1, \theta^2)$ can receive either 1 or 0. By the monotonicity of V we have that $V(\theta^1, \theta^2) > t(1, \theta^1) - t(0, \theta^1)$ if and only if $\theta^2 > C(\theta^1)$. Therefore, if $\theta^2 > C(\theta^1)$

then $q(\theta^1,\theta^2)=1$. By the monotonicity of V we have that $V(\theta^1,\theta^2) < t(1,\theta^1) - t(0,\theta^1)$ if and only if $\theta^2 < C(\theta^1)$. Therefore, if $\theta^2 < C(\theta^1)$ then $q(\theta^1,\theta^2)=0$. ■

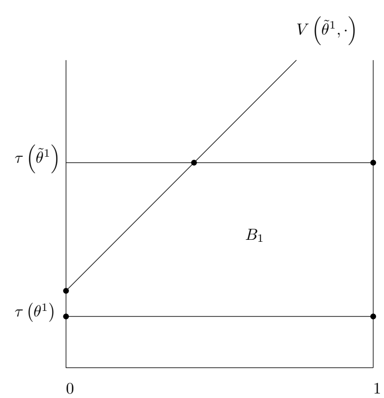
We now proceed to prove that given that the condition in Lemma A is satisfied, it is necessary and sufficient for implementation that $V(\theta^1, C(\theta^1)) := \tau(\theta^1)$ is a decreasing function of θ^1 in the segment $\left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]$.

Lemma B: Assume that $q(\theta)$ is implementable by a sequential mechanism then for every $\theta^1, \tilde{\theta}^1 \in \left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]$ such that $\theta^1 < \tilde{\theta}^1$ we have that $\tau\left(\tilde{\theta}^1\right) \leq \tau(\theta^1)$

Proof: The decision rule $q(\theta)$ is implementable and therefore the condition in Lemma A is satisfied. This means that the report of $\theta^1 \in \left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]$ simply sets $p(\theta^1)$ and $C(\theta^1)$, which in turn set the difference $t(1,\theta^1) - t(0,\theta^1) = V(\theta^1, C(\theta^1))$.
Consider a pair of signals $\theta^1, \tilde{\theta}^1 \in \left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]$ such that $\theta^1 < \tilde{\theta}^1$. Assume that $\tau(\theta^1) < \tau\left(\tilde{\theta}^1\right)$. We consider the following cases:
Case 1: $\tau(\theta^1) \leq V\left(\tilde{\theta}^1, 0\right)$. In this case, assuming the same participation fee for both signals, if $\tilde{\theta}^1$ deviates to θ^1 then he gains

$$\int_0^{V^{-1}\left(\tilde{\theta}^1,\tau\left(\tilde{\theta}^1\right)\right)} V\left(\tilde{\theta}^1, s\right) - \tau\left(\theta^1\right)\, ds + \int_{V^{-1}\left(\tilde{\theta}^1,\tau\left(\tilde{\theta}^1\right)\right)}^1 \tau\left(\tilde{\theta}^1\right) - \tau\left(\theta^1\right)\, ds := B_1$$

therefore it must be that the difference in the participation fees is larger than this gain, i.e., $p(\theta^1) - p\left(\tilde{\theta}^1\right) \geq B_1$.

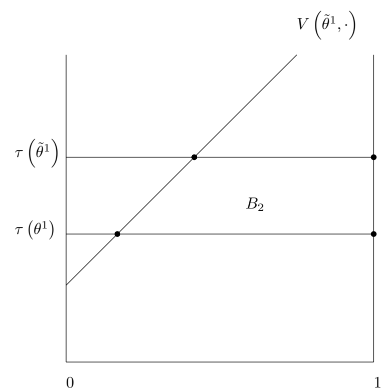


Case 2: $V\left(\tilde{\theta}^1, 0\right) < \tau(\theta^1)$. In this case, assuming the same participation fee for both

signals, if $\tilde{\theta}^1$ deviates to θ^1 then he gains

$$\int\limits_{V^{-1}\left(\tilde{\theta}^1,\tau\left(\theta^1\right)\right)}^{V^{-1}\left(\tilde{\theta}^1,\tau\left(\tilde{\theta}^1\right)\right)} V\left(\tilde{\theta}^1,s\right)-\tau\left(\theta^1\right)\,ds+\int\limits_{V^{-1}\left(\tilde{\theta}^1,\tau\left(\tilde{\theta}^1\right)\right)}^{1} \tau\left(\tilde{\theta}^1\right)-\tau\left(\theta^1\right)\,ds:=B_2$$

therefore it must be that the difference in the participation fees is larger than this gain, i.e., $p\left(\theta^1\right)-p\left(\tilde{\theta}^1\right)\geq B_2$.



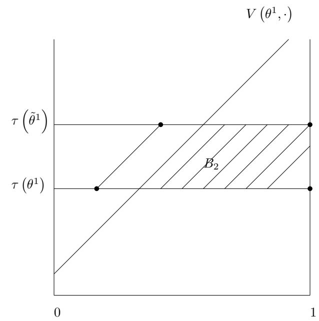
Now for type θ^1 we consider the following possible cases:
Case 1: $V\left(\theta^1,1\right)\leq\tau\left(\tilde{\theta}^1\right)$. In this case if θ^1 deviates to $\tilde{\theta}^1$ then he loses

$$\int\limits_{V^{-1}(\theta^1,\tau(\theta^1))}^{1} V\left(\theta^1,s\right)-\tau\left(\theta^1\right)\,ds<B_1,B_2$$

and therefore there exists a profitable deviation from truthtelling.
Case 2: $\tau\left(\tilde{\theta}^1\right)<V\left(\theta^1,1\right)$. In this case if θ^1case if θ^1 deviates to $\tilde{\theta}^1$ then he loses

$$\int\limits_{V^{-1}(\theta^1,\tau(\theta^1))}^{V^{-1}\left(\theta^1,\tau\left(\tilde{\theta}^1\right)\right)} V\left(\theta^1,s\right)-\tau\left(\theta^1\right)\,ds+\int\limits_{V^{-1}\left(\theta^1,\tau\left(\tilde{\theta}^1\right)\right)}^{1} \tau\left(\tilde{\theta}^1\right)-\tau\left(\theta^1\right)\,ds<B_1,B_2$$

and therefore there exists a profitable deviation from truthtelling.



The loss from deviation is the area marked by the solid lines while the gain from the deviation is the area marked by the black dots

We now prove sufficiency.

Lemma C: Assume that the condition in Lemma A is satisfied and assume that $\tau(\theta^1)$ is decreasing in the segment $\left[\underline{\theta}^{1,C},\overline{\theta}^{1,C}\right]$; then $q(\theta)$ is implementable by a sequential mechanism.

Proof: We define A to be the set of all $\theta^2\in[0,1]$ that are pivotal with respect to some $\theta^1\in\left[\underline{\theta}^{1,C},\overline{\theta}^{1,C}\right]$, i.e.,

$$A\coloneqq\left\{\text{All the }\theta^2\in[0,1]\text{ s.t. there exists a }\theta^1\in\left[\underline{\theta}^{1,C},\overline{\theta}^{1,C}\right]\text{ s.t. }C\left(\theta^1\right)=\theta^2\right\}$$

Now since $\tau(\theta^1)\coloneqq V(\theta^1,C(\theta^1))$ is decreasing we have that $C(\theta^1)$ is strictly decreasing and therefore the function $C^{-1}:A\to[0,1]$ is defined and decreasing.

We define a function $g:[0,1]\setminus A\to A$ that maps every θ^2 that is not pivotal to the following pivotal element,

$$g\left(\theta^2\right)\coloneqq\begin{cases}\sup\left\{\tilde{\theta}^2\text{ s.t. }\tilde{\theta}^2<\theta^2\text{ and }\tilde{\theta}^2\in A\right\} & \text{if }\sup\left\{\tilde{\theta}^2\text{ s.t. }\tilde{\theta}^2<\theta^2\text{ and }\tilde{\theta}^2\in A\right\}\in A\\ \inf\left\{\tilde{\theta}^2\text{ s.t. }\tilde{\theta}^2>\theta^2\text{ and }\tilde{\theta}^2\in A\right\} & \text{if }\inf\left\{\tilde{\theta}^2\text{ s.t. }\tilde{\theta}^2>\theta^2\text{ and }\tilde{\theta}^2\in A\right\}\in A\end{cases}$$

We define another function $\mathring{\theta}^1:[0,1]\to[0,1]$ as follows:

$$\mathring{\theta}^1\left(\theta^2\right)\coloneqq\begin{cases}C^{-1}(\theta^2) & \text{if }\theta^2\in A\\ C^{-1}(g(\theta^2)) & \text{if }\theta^2\notin A\end{cases}$$

We define another function $m : [0,1] \to \mathbb{R}$ as follows:

$$m\left(\theta^{2}\right) \coloneqq V\left(\mathring{\theta}^{1}\left(\theta^{2}\right), \theta^{2}\right)$$

The function m attaches to every pivotal type $\theta^2 \in A$ the second-period price that is received by every agent who reported type $C^{-1}(\theta^2)$ in the first period. Since C^{-1} is decreasing and since lower prices are assigned to higher types of the first period, we get that m is an increasing function. Moreover, note that for every $\tilde{\theta}^1 \in \left[\underline{\theta}^{1,C}, \overline{\theta}^{1,C}\right]$ we have

$$m\left(\theta^{2}\right) = V\left(\tilde{\theta}^{1}, \theta^{2}\right) \text{ at } C\left(\tilde{\theta}^{1}\right) = \theta^{2}$$

$$m\left(\theta^{2}\right) \leq V\left(\tilde{\theta}^{1}, \theta^{2}\right) \text{ for every } C\left(\tilde{\theta}^{1}\right) \leq \theta^{2}$$

$$m\left(\theta^{2}\right) \geq V\left(\tilde{\theta}^{1}, \theta^{2}\right) \text{ for every } C\left(\tilde{\theta}^{1}\right) \geq \theta^{2}$$

We define for every $y \in \text{Image } \tau$:

$$\Theta^{1}(y) = \left\{\theta^{1} \text{ such that } \tau\left(\theta^{1}\right) = y\right\}$$

That is, $\Theta^1(y)$ is the set of signals of the first period that map the price y. For every $y \in \text{Image } \tau$ we set[35] $\underline{\theta}^1(y) \coloneqq \min \Theta^1(y)$ and $C(y) \coloneqq C\left(\underline{\theta}^1(y)\right)$. That is, $C(y)$ assigns to every price y the pivotal second-period type of the lowest first-period type that receives the price y. Let $y' < y$; we get that for every $\theta^1 \in \Theta^1(y)$ we have $C(y') < C(\theta^1) \leq C(y)$.

We define the payment for the option of buying the product in the next period at price y as follows:

$$p(y) = \int_{C(y)}^{1} \left(m(s) - y\right) \, ds$$

We show that under this payment scheme for every $y \in \text{Image } \tau$, every type $\theta^1 \in \Theta^1(y)$ will choose to report truthfully and buy the option of price y for a payment of $p(y)$.

Let $y \in \text{Image } \tau$ and $\theta^1 \in \Theta^1(y)$ assume that this type reports $\hat{\theta}^1 \in \Theta^1(y')$ for $y < y'$;

[35] We assume w.l.o.g. that if $\Theta^1(y)$ is an interval then it is a half closed interval that has a minimum element.

then he loses

$$\int_{C(\theta^1)}^{V^{-1}\left(\theta^1,y'\right)} \left(V\left(\theta_1,s\right)-y\right)\,ds+\int_{V^{-1}\left(\theta^1,y'\right)}^{1}\left(y'-y\right)\,ds\equiv\mathcal{L}$$

and he gains

$$\int_{C(y)}^{1}\left(m\left(s\right)-y\right)\,ds-\int_{C(y')}^{1}\left(m\left(s\right)-y'\right)\,ds=$$

$$\int_{C(y)}^{C(y')}\left(m\left(s\right)-y\right)\,ds+\int_{C(y')}^{1}\left(y'-y\right)\,ds\equiv\mathcal{G}$$

Now $m\left(C\left(y'\right)\right)=y'$, and since m is increasing we get that $m(s)\leq y'$ for every $s<C\left(y'\right)$; in particular, for every $s\in\left[V^{-1}\left(\theta^1,y'\right),C\left(y'\right)\right]$ and therefore

$$\int_{V^{-1}\left(\theta^1,y'\right)}^{C(y')}\left(m\left(s\right)-y\right)\,ds+\int_{C(y')}^{1}\left(y'-y\right)\,ds\leq\int_{V^{-1}\left(\theta^1,y'\right)}^{1}\left(y'-y\right)\,ds$$

In addition, we have that $C\left(\theta^1\right)\leq C\left(y\right)$. We also have that $m\left(s\right)\leq V\left(\theta^1,s\right)$ for every $C\left(\theta^1\right)\leq s$ and therefore

$$\int_{C(y)}^{V^{-1}\left(\theta^1,y'\right)}\left(m\left(s\right)-y\right)\,ds\leq\int_{C(\theta^1)}^{V^{-1}\left(\theta^1,y'\right)}\left(V\left(\theta^1,s\right)-y\right)\,ds$$

and we conclude that $\mathcal{G}\leq\mathcal{L}$. Therefore such a deviation is not profitable.

Let $y\in$ Image τ and $\theta^1\in\Theta^1\left(y\right)$ and assume that this type reports $\hat{\theta}^1\in\Theta^1\left(y'\right)$ for $y'<y$; then he gains

$$\int_{V^{-1}\left(\theta^1,y'\right)}^{C\left(\theta^1\right)}\left(V\left(\theta^1,s\right)-y'\right)\,ds+\int_{C(\theta^1)}^{1}\left(y-y'\right)\,ds\equiv\mathcal{G}$$

and he loses

$$\int_{C(y')}^{1}\left(m\left(s\right)-y'\right)\,ds-\int_{C(y)}^{1}\left(m\left(s\right)-y\right)\,ds=$$

$$\int_{C(y')}^{C(y)}\left(m\left(s\right)-y'\right)\,ds+\int_{C(y)}^{1}\left(y-y'\right)\,ds\equiv\mathcal{L}$$

Now $m\left(C\left(\theta^1\right)\right)=y$ and since m is increasing we get that $m(s)\geq y$ for every $s>C\left(\theta^1\right)$;

in particular for every $s \in [C(\theta^1), C(y)]$ and therefore

$$\int_{C(\theta^1)}^{C(y)} (m(s) - y')\ ds + \int_{C(y)}^{1} (y - y')\ ds \geq \int_{C(\theta^1)}^{1} (y - y')\ ds$$

In addition we have that $C(y') \leq V^{-1}(\theta^1, y')$. We also have that $m(s) \geq V(\theta^1, s)$ for every $C(\theta^1) \geq s$ and therefore

$$\int_{C(y')}^{C\left(\theta^1\right)} (m(s) - y')\ ds \geq \int_{V^{-1}(\theta^1, y')}^{C\left(\theta^1\right)} \left(V\left(\theta^1, s\right) - y'\right)\ ds$$

and we conclude that $\mathcal{L} \geq \mathcal{G}$. Therefore such a deviation is not profitable.

Now, consider the case where $0 < \underline{\theta}^{1,C}$ for every $\theta^1 \leq \underline{\theta}^{1,C}$ we set $p(\theta^1) = 0$. Since $\underline{\theta}^{1,C}$ does not want to deviate to buy an option with a lower price than so does every $\theta^1 < \underline{\theta}^{1,C}$. Consider $\overline{\theta}^{1,C} < \theta^1$ we set $\tau(\theta^1) = V\left(\overline{\theta}^{1,C}, 0\right)$ and $p(\theta^1) = p\left(\overline{\theta}^{1,C}\right)$. Since $\overline{\theta}^{1,C}$ does not want to deviate to buy an option with a higher price than so does every $\overline{\theta}^{1,C} < \theta^1$. ■

Proof of Theorem 7

We show that there exists a sequential second-best mechanism in which there exists $\tilde{\theta}^1 \in [\underline{u}, \overline{u}]$ for which $\tau\left(\tilde{\theta}^1\right) = V\left(\tilde{\theta}^1, \tilde{\theta}^2\left(\tilde{\theta}^1\right)\right)$. That is, the boundary of the second-best decision rule intersects with the boundary of the efficient decision rule. Consider an arbitrary mechanism.

Assume that in this mechanism there is no θ^1 such that $\tau(\theta^1) = V\left(\theta_1, \tilde{\theta}_2(\theta^1)\right)$ and divide it into the four following cases:

(a) $\tau(\theta^1) > V\left(\theta^1, \tilde{\theta}_2(\theta^1)\right)$ for every $\theta^1 \in [\underline{u}, \overline{u}]$. In this case we have that $\tau(\overline{u}) > V\left(\overline{u}, \tilde{\theta}_2(\overline{u})\right)$. Consider a different mechanism that is characterized by the following second-period prices.

$$\tau'(\theta_1) := \begin{cases} \tau(\theta^1) & \text{if } \theta^1 < \overline{u} \\ V\left(\overline{u}, \tilde{\theta}_2(\overline{u})\right) & \text{if } \theta^1 \geq \overline{u} \end{cases}$$

Now $\tau'(\theta_1)$ is a pricing function that satisfies the conditions of Theorem 2: whether

$\overline{u} < 1$ and $\tilde{\theta}_2(\overline{u}) = 0$ or $\overline{u} = 1$ and $\tilde{\theta}_2(\overline{u}) \geq 0$, the function $\tau'(\theta_1)$ is a decreasing function in the areas where the threshold type in the new mechanism, $C'(\theta^1)$, is between 0 and 1. Therefore this mechanism is sequentially implementable and, in addition, it yields an expected social welfare that is at least as high as the expected social welfare in the original mechanism.

(b) $\tau(\theta^1) > V\left(\theta^1, \tilde{\theta}_2(\theta^1)\right)$ for every $\theta^1 \in [\underline{u}, \overline{u}]$. In this case we have that $\tau(\overline{u}) > V\left(\overline{u}, \tilde{\theta}_2(\overline{u})\right)$. Consider a different mechanism that is characterized by the following second-period prices:

$$\tau'(\theta_1) := \begin{cases} V\left(\underline{u}, \tilde{\theta}_2(\underline{u})\right) & \text{if } \theta^1 \leq \underline{u} \\ \tau(\theta^1) & \text{if } \theta^1 > \underline{u} \end{cases}$$

Now $\tau'(\theta_1)$ is a pricing function that satisfies the conditions of Theorem 2: whether $\underline{u} > 0$ and $\tilde{\theta}_2(\underline{u}) = 1$ or $\underline{u} = 0$ and $\tilde{\theta}_2(\overline{u}) \leq 1$, the function $\tau'(\theta_1)$ is a decreasing function in the areas where the threshold type in the new mechanism, $C'(\theta^1)$, is between 0 and 1. Therefore this mechanism is sequentially implementable and, in addition, it yields an expected social welfare that is at least as high as the expected social welfare in the original mechanism.

(c) There exists $\hat{\theta}^1 \in (\underline{u}, \overline{u})$ such that $\tau\left(\hat{\theta}^1\right) > V\left(\hat{\theta}^1, \tilde{\theta}^2\left(\hat{\theta}^1\right)\right)$ and for every $\hat{\theta}^1 < \theta^1$ we have that $\tau(\theta^1) < V\left(\theta^1, \tilde{\theta}^2(\theta^1)\right) < V\left(\hat{\theta}^1, \tilde{\theta}^2\left(\hat{\theta}^1\right)\right)$. Lets look at a different mechanism that is characterized by the following second period prices

$$\tau'(\theta_1) := \begin{cases} V\left(\hat{\theta}^1, \tilde{\theta}_2\left(\hat{\theta}^1\right)\right) & \text{if } \theta^1 = \hat{\theta}^1 \\ \tau(\theta^1) & \text{otherwise} \end{cases}$$

Now $\tau'(\theta_1)$ is a pricing function that satisfies the conditions of Theorem 2. Therefore this mechanism is sequentially implementable and, in addition, it yields an expected social welfare that is at least as high as the expected social welfare in the original mechanism.

(d) There exists $\hat{\theta}^1 \in (\underline{u}, \overline{u})$ such that $\tau\left(\hat{\theta}^1\right) < V\left(\hat{\theta}^1, \tilde{\theta}^2\left(\hat{\theta}^1\right)\right)$ and for every $\theta^1 < \hat{\theta}^1$ we have that $V\left(\hat{\theta}^1, \tilde{\theta}^2\left(\hat{\theta}^1\right)\right) < V\left(\theta^1, \tilde{\theta}^2(\theta^1)\right) < \tau(\theta^1)$. Lets look at a mechanism that is characterized by the following second period prices

$$\tau'(\theta_1) := \begin{cases} V\left(\hat{\theta}^1, \tilde{\theta}^2\left(\hat{\theta}^1\right)\right) & \text{if } \theta^1 = \hat{\theta}^1 \\ \tau(\theta^1) & \text{otherwise} \end{cases}$$

Now $\tau'(\theta^1)$ is a pricing function that satisfies the conditions of Theorem 2. Therefore this mechanism is sequentially implementable and, in addition, it yields an expected social welfare that is at least as high as the expected social welfare in the original mechanism.

This means that for any mechanism for which there is no θ^1 such that $\tau(\theta^1) = V\left(\theta_1, \tilde{\theta}_2(\theta^1)\right)$ there exists another mechanism in which there is a θ^1 such that $\tau(\theta^1) = V\left(\theta_1, \tilde{\theta}_2(\theta^1)\right)$ that yields at least the same expected social welfare. This means that there exists a second best mechanism with the property that there is θ^1 such that $\tau(\theta^1) = V\left(\theta_1, \tilde{\theta}_2(\theta^1)\right)$. The rest of the proof appears in the body of the text.

Proof of Theorem 8

We now show the formal proof for the case where (1) holds; the case where (2) holds is proven by a similar argument. First, we denote by $[\underline{v}, \overline{v}]$ the segment of all θ^1 such that there exists θ^2 where $(\theta^1, \theta^2) \in \bar{V}^{SB}$. We define $\mathring{\theta}^2(\theta^1)$ to be the function that assigns to any $\theta^1 \in [\underline{v}, \overline{v}]$ the threshold type he inflicts with respect to θ^2, i.e., $\mathring{\theta}^2(\theta^1) := \theta^2$ s.t. $(\theta^1, \theta^2) \in \bar{V}^{SB}$. Assume that (1) holds and consider some ε such that $\dot{\theta}^1 + \varepsilon < \bar{u}$. Let

$$V' := \max_{\theta^1 \in \left[\dot{\theta}^1 + \varepsilon, \bar{u}\right]} V\left(\theta^1, \tilde{\theta}^2\left(\theta^1\right)\right)$$

and we have that $V' < V\left(\dot{\theta}^1, \tilde{\theta}^2\left(\dot{\theta}^1\right)\right)$. We define $\hat{\theta}^2(\theta^1)$ as follows:

$$\hat{\theta}^2\left(\theta^1\right) = \begin{cases} \theta^2 \text{ s.t. } V_i(\theta^1, \theta^2) = V' & \text{if such } \theta^2 \text{ exists} \\ 0 & \text{otherwise} \end{cases}$$

Define the function $\tilde{C}(\theta^1)$ as follows:

$$\tilde{C}\left(\theta^1\right) = \begin{cases} 1 & \text{if } 0 \le \theta^1 < \underline{v} \\ \mathring{\theta}^2(\theta^1) & \text{if } \underline{v} \le \theta^1 \le \dot{\theta}^1 + \varepsilon \\ \hat{\theta}^2(\theta^1) & \text{if } \dot{\theta}^1 + \varepsilon < \theta^1 \le \overline{u} \\ 0 & \text{if } \overline{u} < \theta^1 \le 1 \end{cases}$$

Consider a decision rule $\tilde{q}(\theta)$ that takes the following form:

$$\tilde{q}(\theta) = \begin{cases} 1 \text{ if} & \theta^2 \ge \tilde{C}(\theta^1) \\ 0 & \text{otherwise} \end{cases}$$

The function $V\left(\theta^1, \tilde{C}(\theta^1)\right)$ is decreasing in the segment $\left[\underline{\theta}^{1,\tilde{C}}, \overline{\theta}^{1,\tilde{C}}\right]$ and therefore $\tilde{q}(\theta)$ is implementable by a sequential mechanism. To see that the social welfare under $\tilde{q}(\theta)$ is greater than under $q^{SB}(\theta)$, note that $q^{SB}(\theta)$ and $\tilde{q}(\theta)$ coincide except for a set of positive measure that lies above the boundary of the efficient decision rule in which $\tilde{q}(\theta)$ allocates the item to the buyer and $q^{SB}(\theta)$ allocates the item to the seller.

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